The measurement process in domain theory

Keye Martin

Department of Mathematics, Tulane University, New Orleans, LA 70118, USA
Fax (504)865-5063
martin@math.tulane.edu
http://www.math.tulane.edu/~martin

Abstract. We introduce the measurement idea in domain theory and then apply it to establish two fixed point theorems. The first is an extension of the Scott fixed point theorem which applies to nonmonotonic mappings. The second is a contraction principle for monotone maps that guarantees the existence of *unique* fixed points.

1 Introduction

A measurement on a domain D is a Scott continuous map $\mu : D \to [0,\infty)^*$ into the nonnegative reals in their *reverse* order which formalizes the notion *information content* for objects in a domain. Intuitively, if $x \in D$ is an *informative* object, then μx is the *amount of information* it contains. In another light, we may think of μ as measuring the disorder in an object, or *entropy,* since $x \sqsubseteq y \Rightarrow \mu x \geq \mu y$, that is, the more informative an object is, the smaller its measure.

After giving a precise definition of measurement and several natural examples, we show the value of the idea by proving and then applying two fixed point theorems. The first is an extension of the Scott fixed point theorem which applies to nonmonotonic processes, like the bisection method and the r-section search. The second is a contraction principle for monotone maps that guarantees the existence of *unique* fixed points, as opposed to the *least* fixed points that domain theory usually provides.

2 Background

2.1 Domain theory

A *poset* is a partially ordered set [1].

Definition 1. A *least element* in a poset $(P, \sqsubseteq)$ is an element $\perp \in P$ such that $\perp \sqsubseteq x$ for all $x \in P$. Such an element is unique. An element $x \in P$ is *maximal* if $(\forall y \in P)\, x \sqsubseteq y \Rightarrow x = y$. The set of maximal elements in a poset is written $\max P$.

Definition 2. Let $(P, \sqsubseteq)$ be a poset. A nonempty subset $S \subseteq P$ is *directed* if $(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z$. The *supremum* of a subset $S \subseteq P$ is the least of all its upper bounds provided it exists. This is written $\bigsqcup S$. A *dcpo* is a poset in which every directed subset has a supremum.

Definition 3. In a poset $(P, \sqsubseteq)$, $a \ll x$ iff for all directed subsets $S \subseteq D$ which have a supremum, $x \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S)\, a \sqsubseteq s$. We set $\mathord{\downarrow\!\!\!\downarrow} x = \{a \in P : a \ll x\}$. An element $x \in P$ is *compact* if $x \ll x$. The set of compact elements in P is $K(P)$.

Definition 4. A subset B of a poset P is a *basis* for P if $B \cap \mathord{\downarrow\!\!\!\downarrow} x$ contains a directed subset with supremum x, for each $x \in P$.

Definition 5. A poset is *continuous* if it has a basis. A poset is *algebraic* if its compact elements form a basis. A poset is *ω-continuous* if it has a countable basis.

Definition 6. A *domain* is a continuous dcpo.

Definition 7. A subset U of a poset P is *Scott open* if

(i) U is an upper set: $x \in U$ & $x \sqsubseteq y \Rightarrow y \in U$, and
(ii) U is inaccessible by directed suprema: For every directed $S \subseteq P$ with a supremum,

$$\bigsqcup S \in U \Rightarrow S \cap U \neq \emptyset.$$

The collection of all Scott open subsets of P is called the Scott topology. It is denoted σ_P.

Unless explicitly stated otherwise, all topological statements about posets are made with respect to the Scott topology.

Proposition 1. *A function $f : D \to E$ between dcpos is continuous iff*

(i) *f is monotone: $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$.*
(ii) *f preserves directed suprema: For all directed $S \subseteq D$, $f(\bigsqcup S) = \bigsqcup f(S)$.*

2.2 Examples of domains

Example 1. The *interval domain* is the collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{[a, b] : a, b \in \mathbb{R} \;\&\; a \leq b\}$$

ordered under reverse inclusion

$$[a, b] \sqsubseteq [c, d] \Leftrightarrow [c, d] \subseteq [a, b]$$

is an ω-continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{I}\mathbb{R}$ is $\bigcap S$, while the approximation relation is characterized by $I \ll J \Leftrightarrow J \subseteq \operatorname{int}(I)$. A countable basis for $\mathbf{I}\mathbb{R}$ is given by $\{[p, q] : p, q \in \mathbb{Q} \;\&\; p \leq q\}$.

Definition 8. A *partial function* $f : X \rightharpoonup Y$ between sets X and Y is a function $f : A \to Y$ defined on a subset $A \subseteq X$. We write $\mathrm{dom}(f) = A$ for the *domain* of a partial map $f : X \rightharpoonup Y$.

Example 2. The set of *partial mappings on the naturals*

$$[\mathbb{N} \rightharpoonup \mathbb{N}] = \{ f \mid f : \mathbb{N} \rightharpoonup \mathbb{N} \}$$

becomes an ω-algebraic dcpo when ordered by extension

$$f \sqsubseteq g \Leftrightarrow \mathrm{dom}(f) \subseteq \mathrm{dom}(g) \ \& \ f = g \text{ on } \mathrm{dom}(f).$$

The supremum of a directed set $S \subseteq [\mathbb{N} \to \mathbb{N}]$ is $\bigcup S$, under the view that functions are certain subsets of $\mathbb{N} \times \mathbb{N}$, while the approximation relation is

$$f \ll g \Leftrightarrow f \sqsubseteq g \ \& \ \mathrm{dom}(f) \text{ is finite}.$$

The maximal elements of $[\mathbb{N} \to \mathbb{N}]$ are the *total functions*, that is, those functions f with $\mathrm{dom}(f) = \mathbb{N}$.

Example 3. The *Cantor set model* is the collection of functions

$$\Sigma^\infty = \{ s \mid s : \{1, \ldots, n\} \to \{0, 1\}, 0 \leq n \leq \infty \}$$

is also an ω-algebraic dcpo under the extension order

$$s \sqsubseteq t \Leftrightarrow |s| \leq |t| \ \& \ (\, \forall\, 1 \leq i \leq |s| \,)\ s(i) = t(i),$$

where $|s|$ is written for the cardinality of $\mathrm{dom}(s)$. The supremum of a directed set $S \subseteq \Sigma^\infty$ is $\bigcup S$, while the approximation relation is

$$s \ll t \Leftrightarrow s \sqsubseteq t \ \& \ |s| < \infty.$$

The extension order in this special case is usually called the *prefix* order. The elements $s \in \Sigma^\infty$ are called *strings* over $\{0, 1\}$. The quantity $|s|$ is called the *length* of a string s. The *empty string* ε is the unique string with length zero. It is the least element $\perp$ of Σ^∞.

Example 4. If X is a locally compact Hausdorff space, then its *upper space*

$$\mathbf{U}X = \{\emptyset \neq K \subseteq X : K \textit{ is compact}\}$$

ordered under reverse inclusion

$$A \sqsubseteq B \Leftrightarrow B \subseteq A$$

is a continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{U}X$ is $\bigcap S$ and the approximation relation is $A \ll B \Leftrightarrow B \subseteq \mathrm{int}(A)$.

Example 5. Given a metric space (X, d), the *formal ball model* [2]

$$\mathbf{B}X = X \times [0, \infty)$$

is a poset when ordered via

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leq r - s.$$

The approximation relation is characterized by

$$(x, r) \ll (y, s) \Leftrightarrow d(x, y) < r - s.$$

The poset $\mathbf{B}X$ is continuous. However, $\mathbf{B}X$ is a dcpo iff the metric d is complete. In addition, $\mathbf{B}X$ has a countable basis iff X is a separable metric space.

3 Measurement

The set $[0, \infty)^*$ is the domain of nonnegative reals in their opposite order.

Definition 9. A Scott continuous map $\mu : D \to [0, \infty)^*$ on a continuous dcpo D *induces the Scott topology near* $X \subseteq D$ if for all Scott open sets $U \subseteq D$ and for any $x \in X$,

$$x \in U \Rightarrow (\,\exists\, \varepsilon > 0\,)\; x \in \mu_\varepsilon(x) \subseteq U,$$

where $\mu_\varepsilon(x) = \{y \in D : y \sqsubseteq x \;\&\; |\mu x - \mu y| < \varepsilon\}$. This is written $\mu \to \sigma_X$.

Definition 10. A *measurement* on a domain D is a Scott continuous mapping $\mu : D \to [0, \infty)^*$ with $\mu \to \sigma_{\ker \mu}$ where $\ker \mu = \{x \in D : \mu x = 0\}$.

The most useful properties of measurements in applications are as follows.

Proposition 2. *If D is a domain with a measurement $\mu \to \sigma_X$, then*

(i) *For all $x \in D$ and $y \in X$, $x \sqsubseteq y \;\&\; \mu x = \mu y \Rightarrow x = y$.*
(ii) *For all $x \in D$, $\mu x = 0 \Rightarrow x \in \max D$.*
(iii) *For all $x \in X$ and any sequence (x_n) in D with $x_n \sqsubseteq x$, if $\mu x_n \to \mu x$, then $\bigsqcup x_n = x$, and this supremum converges in the Scott topology.*

Proof For (i), we prove that $y \sqsubseteq x$. Let U be a Scott open set around y. Then there is $\varepsilon > 0$ with $y \in \mu_\varepsilon(y) \subseteq U$. But $x \sqsubseteq y$ and $\mu x = \mu y$ hence $x \in \mu_\varepsilon(y) \subseteq U$. Thus, every Scott open set around y also contains x, establishing $y \sqsubseteq x$. (ii) follows from (i). The proof of (iii) uses the same technique applied in (i) and may be found in [3]. $\square$

Prop. 2 shows that measurements capture the essential characteristics of information content. For example, (i) says that comparable objects with the same amount of information are equal, (ii) says that an element with no disorder in it (no partiality) must be maximal in the information order, and (iii) says that if we measure an iterative process (x_n) as computing an object x, then it *actually does* calculate x. The common theme in each case is this: Any observation made with a measurement is a *reliable* one.

Example 6. Domains and their standard measurements.

(i) $(\mathbf{I}\mathbb{R}, \mu)$ the interval domain with the length measurement $\mu[a,b] = b - a$.
(ii) $([\mathbb{N} \rightharpoonup \mathbb{N}], \mu)$ the partial functions on the naturals with

$$\mu f = |\mathrm{dom}(f)|$$

where $|\cdot| : \mathcal{P}\omega \to [0,\infty)^*$ is the measurement on the algebraic lattice $\mathcal{P}\omega$ given by

$$|x| = 1 - \sum_{n \in x} \frac{1}{2^{n+1}}.$$

(iii) $(\Sigma^\infty, 1/2^{|\cdot|})$ the Cantor set model where $|\cdot| : \Sigma^\infty \to [0,\infty]$ is the length of a string.
(iv) $(\mathbf{U}X, \mathrm{diam})$ the upper space of a locally compact metric space (X,d) with

$$\mathrm{diam}\, K = \sup\{d(x,y) : x, y \in K\}.$$

(v) $(\mathbf{B}X, \pi)$ the formal ball model of a complete metric space (X,d) with $\pi(x,r) = r$.

In each example above, we have a measurement $\mu : D \to [0,\infty)^*$ on a domain with $\ker\mu = \max D$. In all cases except (iv), we also have $\mu \to \sigma_D$. In general, there are existence theorems [3] for countably based domains, which show that measurements usually exist. However, the value of the idea lies not in knowing that they exist abstractly, but in knowing that *particular* mappings, like the ones in the last example, are measurements.

4 Fixed points of nonmonotonic maps

Definition 11. A *splitting* on a poset P is a function $s : P \to P$ with $x \sqsubseteq s(x)$ for all $x \in P$.

Proposition 3. *Let D be a domain with a measurement $\mu \to \sigma_D$. If $I \subseteq D$ is closed under directed suprema and $s : I \to I$ is a splitting whose measure*

$$\mu \circ s : I \to [0,\infty)^*$$

is Scott continuous between dcpo's, then

$$(\forall x \in I) \bigsqcup_{n \geq 0} s^n(x) \text{ is a fixed point of } s.$$

Moreover, the set of fixed points $\mathrm{fix}(s) = \{x \in I : s(x) = x\}$ *is a dcpo.*

Proof Let $x \in I$. By induction, $(s^n(x))$ is an increasing sequence in I. The set I is closed under directed suprema hence $\bigsqcup_{n \geq 0} s^n(x) \in I$. Because s is a splitting,

$\bigsqcup_{n\geq 0} s^n(x) \sqsubseteq s(\bigsqcup_{n\geq 0} s^n(x))$, while the fact that $\mu \circ s$ and μ are both Scott continuous allows us to compute

$$\mu s(\bigsqcup_{n\geq 0} s^n(x)) = \lim_{n\to\infty} \mu s^{n+1}(x) = \mu(\bigsqcup_{n\geq 0} s^n(x)).$$

By Prop 2, however, two comparable elements whose measures agree must in fact be equal. Hence,

$$s(\bigsqcup_{n\geq 0} s^n(x)) = \bigsqcup_{n\geq 0} s^n(x).$$

To show that $\mathrm{fix}(s)$ is a dcpo one need only prove closure under suprema of sequences because $\mu \to \sigma_D$ [3]. The proof for sequences, however, uses the very same methods employed above and is entirely trivial. ☐

Example 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map on the real line. Denote by $C(f)$ the subset of $\mathbf{I}\mathbb{R}$ where f changes sign, that is,

$$C(f) = \{[a,b] : f(a)\cdot f(b) \leq 0\}.$$

The continuity of f ensures that this set is closed under directed suprema, and the mapping

$$\mathrm{split}_f : C(f) \to C(f)$$

given by

$$\mathrm{split}_f[a,b] = \begin{cases} \mathrm{left}[a,b] & \text{if } \mathrm{left}[a,b] \in C(f); \\ \mathrm{right}[a,b] & \text{otherwise,} \end{cases}$$

is a splitting where $\mathrm{left}[a,b] = [a,(a+b)/2]$ and $\mathrm{right}[a,b] = [(a+b)/2,b]$. The measure of this mapping

$$\mu\, \mathrm{split}_f[a,b] = \frac{\mu[a,b]}{2}$$

is Scott continuous, so Proposition 3 implies that

$$\bigsqcup_{n\geq 0} \mathrm{split}_f^n[a,b] \in \mathrm{fix}(\mathrm{split}_f).$$

However, $\mathrm{fix}(\mathrm{split}_f) = \{[r] : f(r) = 0\}$, which means that iterating split_f is a scheme for calculating a solution of the equation $f(x) = 0$. This numerical technique is called *the bisection method.*

The major fixed point technique in classical domain theory, the Scott fixed point theorem, cannot be used to establish the correctness of the bisection method: split_f is only monotone in computationally irrelevant cases.

Proposition 4. *For a continuous selfmap $f : \mathbb{R} \to \mathbb{R}$ which has at least one zero, the following are equivalent:*

(i) *The map* split_f *is monotone.*
(ii) *The map f has a unique zero r and*

$$C(f) = \{[a, r] : a \leq r\} \cup \{[r, b] : r \leq b\}.$$

Proof We prove (i) $\Rightarrow$ (ii). Let $\alpha < \beta$ be two distinct roots of f. Then by monotonicity of split_f,

$$\text{split}_f^n[\alpha, \beta] \sqsubseteq \text{split}_f[\beta] = [\beta],$$

for all $n \geq 0$. Then $[\alpha] = \bigsqcup \text{split}_f^n[\alpha, \beta] \sqsubseteq [\beta]$, which proves $\alpha = \beta$. Thus, f has a unique zero r.

Now let $[a, b] \in C(f)$ with $a < r < b$ and set $\delta = \max\{r - a, b - r\} > 0$. Then $r - \delta \leq a < b \leq r + \delta$. By the uniqueness of r,

$$f(r - \delta) \cdot f(a) > 0 \text{ and } f(b) \cdot f(r + \delta) > 0,$$

and since $[a, b] \in C(f)$, we have $\bar{y} := [r - \delta, r + \delta] \in C(f)$. For the very same reason, $\bar{x} := [r - \delta - \delta/2, r + \delta + \delta/4] \in C(f)$. But then we have $\bar{x} \sqsubseteq \bar{y}$ and

$$\text{split}_f \bar{x} = [r - \delta/8, r + \delta + \delta/4] \not\sqsubseteq [r - \delta, r] = \text{split}_f \bar{y},$$

which means split_f is not monotone if f changes sign on an interval which contains r in its interior. ☐

That is, if split_f is monotone, then in order to calculate the solution r of $f(x) = 0$ using the bisection method, we must first *know* the solution r.

Example 8. A function $f : [a, b] \to \mathbb{R}$ is *unimodal* if it has a maximum value assumed at a unique point $x^* \in [a, b]$ such that

(i) f is strictly increasing on $[a, x^*]$, and
(ii) f is strictly decreasing on $[x^*, b]$.

Unimodal functions have the important property that

$$x_1 < x_2 \Rightarrow \begin{cases} x_1 \leq x^* \leq b \text{ if } f(x_1) < f(x_2), \\ a \leq x^* \leq x_2 \text{ otherwise.} \end{cases}$$

This observation leads to an algorithm for computing x^*. For a unimodal map $f : [a, b] \to \mathbb{R}$ with maximizer $x^\star \in [a, b]$ and a constant $1/2 < r < 1$, define a dcpo by

$$I_{x^*} = \{\bar{x} \in \mathbf{I}\mathbb{R} : [a, b] \sqsubseteq \bar{x} \sqsubseteq [x^*]\},$$

and a splitting by

$$\max_f : I_{x^*} \to I_{x^*}$$

$$\max_f[a, b] = \begin{cases} [l(a, b), b] \;\text{ if } f(l(a, b)) < f(r(a, b)); \\ [a, r(a, b)] \text{ otherwise,} \end{cases}$$

where $l(a,b) = (b-a)(1-r)+a$ and $r(a,b) = (b-a)r+a$. The measure of $\max_f$ is Scott continuous since $\mu \max_f(\bar{x}) = r \cdot \mu(\bar{x})$, for all $\bar{x} \in I_{x^*}$. By Proposition 3,

$$\bigsqcup_{n\geq 0} \max_f^n(\bar{x}) \in \mathrm{fix}(\max_f),$$

for any $\bar{x} \in I_{x^*}$. However, any fixed point of $\max_f$ has measure zero, and the only element of I_{x^*} with measure zero is $[x^*]$. Thus, $\bigsqcup \max_f^n[a,b] = [x^*]$, which means that iterating $\max_f$ yields a method for calculating x^*. This technique is called the *r-section search.*

Finally, observe that $\max_f$ is not monotone. Let -1 $< \alpha < 1$ and $f(x) = 1-x^2$. The function f is unimodal on any compact interval. Since $\max_f[\text{-}1,1] = [\text{-}1, 2r-1]$, we see that

$$\begin{aligned}\max_f[\text{-}1,1] \sqsubseteq \max_f[\alpha,1] &\Rightarrow 1 \leq 2r-1 \text{ or } r(\alpha,1) \leq 2r-1\\ &\Rightarrow 1 \leq r \text{ or } \alpha+1 \leq r(\alpha+1)\\ &\Rightarrow r \geq 1,\end{aligned}$$

which contradicts $r < 1$. Thus, for no value of r is the algorithm monotone.

As further evidence of its applicability, notice that Prop. 3 also implies the Scott fixed point theorem for domains with measurements $\mu \to \sigma_D$.

Example 9. If $f : D \to D$ is a Scott continuous map on a domain D with a measurement $\mu \to \sigma_D$, then we consider its restriction to the set of points where it improves

$$I(f) = \{x \in D : x \sqsubseteq f(x)\}.$$

This evidently yields a splitting $f : I(f) \to I(f)$ on a dcpo with continuous measure. By Proposition 3,

$$(\forall x \in I(f)) \bigsqcup_{n\geq 0} f^n(x) \text{ is a fixed point of } f.$$

For instance, if D is ω-continuous with basis $\{b_n : n \in \mathbb{N}\}$, then

$$\mu x = |\{n : b_n \ll x\}|$$

defines a measurement $\mu \to \sigma_D$. Notice, however, that with this construction we normally have $\ker \mu = \emptyset$.

5 Unique fixed points of monotonic maps

In the last section, we saw that measurement can be used to generalize the Scott fixed point theorem so as to include important nonmonotonic processes. Now we see that in can improve upon it for monotone maps as well, by giving a technique that guarantees *unique* fixed points.

Definition 12. Let D be a continuous dcpo with a measurement μ. A monotone map $f : D \to D$ is a *contraction* if there is a constant $c < 1$ with

$$\mu f(x) \leq c \cdot \mu x$$

for all $x \in D$.

Theorem 1. *Let D be a domain with a measurement μ such that*

$$(\,\forall\, x, y \in \ker \mu\,)(\,\exists\, z \in D\,)\; z \sqsubseteq x, y.$$

If $f : D \to D$ is a contraction and there is a point $x \in D$ with $x \sqsubseteq f(x)$, then

$$x^\star = \bigsqcup_{n \geq 0} f^n(x) \in \max D$$

is the unique fixed point of f on D. Furthermore, $x^\star$ is an attractor in two different senses:

(i) *For all $x \in \ker \mu$, $f^n(x) \to x^\star$ in the Scott topology on $\ker \mu$, and*
(ii) *For all $x \sqsubseteq x^\star$, $\bigsqcup_{n \geq 0} f^n(x) = x^\star$, and this supremum is a limit in the Scott topology on D.*

Proof First, for any $x \in D$ and any $n \geq 0$, $\mu f^n(x) \leq c^n \mu x$, as is easy to prove by induction. Given a point $x \sqsubseteq f(x)$, the monotonicity of f implies the sequence $(f^n(x))$ is increasing, while the continuity of μ allows us to compute

$$\mu(\bigsqcup f^n(x)) = \lim_{n \to \infty} \mu f^n(x) \leq \lim_{n \to \infty} c^n \mu x = 0.$$

Hence, $x^\star = \bigsqcup f^n(x) \in \ker \mu \subseteq \max D$. But the monotonicity of f also gives $x^\star \sqsubseteq f(x^\star)$. Hence, $x^\star = f(x^\star)$ is a fixed point of f. We will prove its uniqueness after (ii).

For (ii), let $x \sqsubseteq x^\star$. By the monotonicity of f,

$$(\forall n \geq 0)\; f^n(x) \sqsubseteq f(x^\star) = x^\star,$$

and since $\lim \mu f^n(x) = \mu x^\star = 0$, the fact that μ is a measurement yields

$$\bigsqcup_{n \geq 0} f^n(x) = x^\star.$$

Now let $x_\star$ be any fixed point of f. Then $x_\star \in \ker \mu$ so there is $z \in D$ with $z \sqsubseteq x_\star, x^\star$. By (ii), $\bigsqcup f^n(z) = x_\star = x^\star$. Thus, the fixed point $x^\star$ is unique.

For (i), let $x \in \ker \mu$. Then $f^n(x) \in \ker \mu$ for all $n \geq 0$. In addition, there is an $a \sqsubseteq x, x^\star$, so $f^n(a) \sqsubseteq f^n(x), x^\star$. Now let U be a Scott open set around x^*. Because μ is a measurement,

$$(\exists\, \varepsilon > 0)\; x^\star \in \mu_\varepsilon(x^\star) \subseteq U.$$

Since $\mu f^n(a) \to 0$, all but a finite number of the $f^n(a)$ are in U. But U is an upper set, so the same is true of the $f^n(x)$. Hence, $f^n(x) \to x^\star$, in the Scott topology on $\ker \mu$. □

When a domain has a least element, the last result is easier to state.

Corollary 1. *Let D be a domain with least element $\perp$ and measurement μ. If $f : D \to D$ is a contraction, then*

$$x^\star = \bigsqcup_{n\geq 0} f^n(\perp) \in \max D$$

is the unique fixed point of f on D. In addition, the other conclusions of Theorem 1 hold as well.

All the domains that we have considered in this paper have the property that $(\forall x, y \in D)(\exists z \in D)\, z \sqsubseteq x, y$, and so Theorem 1 can be applied to them.

Example 10. Let $f : X \to X$ be a contraction on a complete metric space X with Lipschitz constant $c < 1$. The mapping $f : X \to X$ extends to a monotone map on the formal ball model $\bar{f} : \mathbf{B}X \to \mathbf{B}X$ given by

$$\bar{f}(x, r) = (fx, c \cdot r),$$

which satisfies

$$\pi\bar{f}(x, r) = c \cdot \pi(x, r),$$

where $\pi : \mathbf{B}X \to [0, \infty)^*$ is the standard measurement on $\mathbf{B}X$, $\pi(x, r) = r$. Now choose r so that $(x, r) \sqsubseteq \bar{f}(x, r)$. By Theorem 1, $\bar{f}$ has a unique attractor which implies that f does also because $X \simeq \ker \pi$.

We can also use the upper space $(\mathbf{U}X, \mathrm{diam})$ to prove the Banach contraction theorem for compact metric spaces by applying the technique of the last example.

Example 11. Consider the well-known functional

$$\phi : [\mathbb{N} \rightharpoonup \mathbb{N}] \to [\mathbb{N} \rightharpoonup \mathbb{N}]$$

$$\phi(f)(k) = \begin{cases} 1 & \text{if } k = 0, \\ kf(k-1) & \text{if } k \geq 1 \;\&\; k-1 \in \mathrm{dom}\, f. \end{cases}$$

which is easily seen to be monotone. Applying $\mu : [\mathbb{N} \rightharpoonup \mathbb{N}] \to [0, \infty)^*$, we compute

$$\begin{aligned}
\mu\phi(f) &= |\mathrm{dom}(\phi(f))| \\
&= 1 - \sum_{k\in\mathrm{dom}(\phi(f))} \frac{1}{2^{k+1}} \\
&= 1 - \left(\frac{1}{2^{0+1}} + \sum_{k-1\in\mathrm{dom}(f)} \frac{1}{2^{k+1}} \right) \\
&= 1 - \left(\frac{1}{2} + \sum_{k\in\mathrm{dom}(f)} \frac{1}{2^{k+2}} \right) \\
&= \frac{1}{2} \left(1 - \sum_{k\in\mathrm{dom}(f)} \frac{1}{2^{k+1}} \right) \\
&= \frac{\mu f}{2}
\end{aligned}$$

which means ϕ is a contraction on the domain $[\mathbb{N} \rightharpoonup \mathbb{N}]$. By the contraction principle,

$$\bigsqcup_{n\in\mathbb{N}} \phi^n(\perp) = \mathrm{fac}$$

is the unique fixed point of ϕ on $[\mathbb{N} \rightharpoonup \mathbb{N}]$, where $\perp$ is the function defined nowhere.

One wonders here about the potential for replacing metric space semantics with an approach based on measurement and contractions.

6 Closing remarks

There are many ideas left from the present discussion on measurement. Among the most fundamental are the μ topology, the study of the topological structure of $\ker\mu$, a discussion of how one extends measurements to higher order domains, and the informatic derivative (the derivative of a map on a domain with respect to a measurement). All of this can be found on the author's webpage in [3].

References

1. S. Abramsky and A. Jung. *Domain Theory.* In S. Abramsky, D. M. Gabbay, T. S. E. Maibaum, editors, Handbook of Logic in Computer Science, vol. III. Oxford University Press, 1994
2. A. Edalat and R. Heckmann. *A Computational Model for Metric Spaces.* Theoretical Computer Science 193 (1998) 53–73 .
3. K. Martin. *A foundation for computation.* Ph.D. thesis, Department of Mathematics, Tulane University, 2000.