A Foundation for Computation

Keye Martin

April 14, 2004

Contents

Chapter 1

Introduction

1.1 Domain Theory

1.1.1 Order

The reader unfamiliar with the basics of Domain theory will find [1] valuable. We touch on certain basic aspects which are not quoted very often.

Definition 1.1.1 A partially ordered set $(P, \sqsubseteq)$ is a set P together with a binary relation $\sqsubseteq \subseteq P^2$ which is

(i) reflexive: $(\forall x \in P)\, x \sqsubseteq x$,

(ii) antisymmetric: $(\forall x, y \in P)\, x \sqsubseteq y \;\&\; y \sqsubseteq x \Rightarrow x = y$, *and*

(iii) transitive: $(\forall x, y, z \in P)\, x \sqsubseteq y \;\&\; y \sqsubseteq z \Rightarrow x \sqsubseteq z$.

We refer to partially ordered sets as *posets.*

Definition 1.1.2 A *least element* in a poset $(P, \sqsubseteq)$ is an element $\perp \in P$ such that $\perp \sqsubseteq x$ for all $x \in P$. Such an element is unique and is called a *bottom* element. An element $x \in P$ is *maximal* if $(\forall y \in P)\, x \sqsubseteq y \Rightarrow x = y$. The set of maximal elements in a poset is written $\max P$. Similarly, one has the notions of *greatest element* and *minimal element.*

Definition 1.1.3 For a subset X of a poset $(P, \sqsubseteq)$, let

$$\uparrow\! X = \{y \in P : (\exists x \in X)\, x \sqsubseteq y\} \;\&\; \downarrow\! X = \{y \in P : (\exists x \in X)\, y \sqsubseteq x\}.$$

We say that X is an *upper set* if $X = \uparrow\! X$ and a *lower set* if $X = \downarrow\! X$.

Definition 1.1.4 Let $(P, \sqsubseteq)$ be a poset. A nonempty subset $S \subseteq P$ is *directed* if $(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z$. The *supremum* of a subset $S \subseteq P$ is the least of all its upper bounds provided it exists. This is written $\bigsqcup S$.

A poset $(P, \sqsubseteq)$ is abbreviated to P, just as in topology, where one writes X for the topological space (X, τ).

Definition 1.1.5 In a poset $(P, \sqsubseteq)$, $a \ll x$ iff for all directed subsets $S \subseteq P$ which have a supremum,

$$x \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S)\, a \sqsubseteq s.$$

We say that a is an approximation of x whenever $a \ll x$. The set of all approximations of x is written $\twoheaddownarrow x$. An element $x \in P$ is *compact* if $x \ll x$. The set of compact elements in a poset P is written $K(P)$.

Definition 1.1.6 A poset P is *continuous* if $\twoheaddownarrow x$ is directed with supremum x for all $x \in P$.

It is usually easier to find a basis for a poset.

Definition 1.1.7 A subset B of a poset P is a *basis* for P if $B \cap \twoheaddownarrow x$ contains a directed subset with supremum x, for each $x \in P$.

Lemma 1.1.1 *A poset is continuous iff it has a basis.*

Definition 1.1.8 A poset is *algebraic* if its compact elements form a basis. A poset is ω-*continuous* if it has a countable basis.

Continuity provides a definite notion of *approximation* for posets.

Proposition 1.1.1 (Zhang [20]) *Continuous posets have the interpolation property:* $x \ll y \Rightarrow (\exists z)\, x \ll z \ll y$.

A useful form of *completeness* is offered by a dcpo.

Definition 1.1.9 A poset is a *dcpo* if every directed subset has a supremum.

Domains possess both approximation and completeness.

Definition 1.1.10 A *domain* is a continuous poset which is also a dcpo. A domain is also called a *continuous dcpo.*

1.1.2 The Topological Aspect

One of the interesting things about a domain is that its order-theoretic structure is rich enough to support the derivation of intrinsically defined topologies. The most well-known of these is the Scott topology.

Definition 1.1.11 A subset U of a poset P is *Scott open* if

(i) U is an upper set: $x \in U \;\&\; x \sqsubseteq y \Rightarrow y \in U$, and

(ii) U is inaccessible by directed suprema: For every directed $S \subseteq P$ which has a supremum,
$$\bigsqcup S \in U \Rightarrow S \cap U \neq \emptyset.$$

The collection of all Scott open sets on P is called the Scott topology. It is denoted σ_P.

Unless explicitly stated otherwise, all topological statements about posets are made with respect to the Scott topology.

Proposition 1.1.2 *A function $f : P \to Q$ between posets is continuous iff*

(i) *f is monotone: $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$.*

(ii) *f preserves directed suprema: For every directed $S \subseteq P$ which has a supremum,*
$$\bigsqcup f(S) \text{ exists } \& \; f(\bigsqcup S) = \bigsqcup f(S).$$

Proposition 1.1.3 (Zhang [20]) *The collection $\{ \twoheaduparrow x : x \in P \}$ is a basis for the Scott topology on a continuous poset P.*

One of the most important topological results concerning the Scott topology on a domain is the following.

Theorem 1.1.1 (Hofmann-Mislove) *In a continuous dcpo, the class of nonempty compact upper sets is closed under filtered intersections.*

This is called the *Hofmann-Mislove* theorem.

Definition 1.1.12 The *Lawson topology* on a continuous poset P has as a basis all sets of the form $\twoheaduparrow x \setminus \uparrow F$ where $x \in P$ and $F \subseteq P$ is finite.

Proposition 1.1.4 (Jung [13]) *The Lawson topology on a continuous dcpo is compact iff it is Scott compact and the intersection of any two Scott compact upper sets is Scott compact.*

Definition 1.1.13 A *Scott domain* is a continuous dcpo with least element $\perp$ in which each pair of elements bounded from above has a supremum.

Notice that we have not required Scott domains to be ω-algebraic.

Proposition 1.1.5 *Every Scott domain has compact Lawson topology.*

1.2 Examples of Domains

This section discusses some domains we will refer to on a regular basis.

1.2.1 The Interval Domain

The collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{[a, b] : a, b \in \mathbb{R} \ \& \ a \leq b\}$$

ordered under reverse inclusion

$$[a, b] \sqsubseteq [c, d] \Leftrightarrow [c, d] \subseteq [a, b]$$

is an ω-continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{I}\mathbb{R}$ is $\bigcap S$, while the approximation relation is characterized by $I \ll J \Leftrightarrow J \subseteq \text{int}(I)$. A countable basis for $\mathbf{I}\mathbb{R}$ is given by $\{[p, q] : p, q \in \mathbb{Q} \ \& \ p \leq q\}$.

1.2.2 The Powerset of the Naturals

The collection of subsets

$$\mathcal{P}\omega = \{x : x \subseteq \mathbb{N}\}$$

ordered under inclusion

$$x \sqsubseteq y \Leftrightarrow x \subseteq y$$

is an ω-algebraic dcpo. The supremum of a directed set $S \subseteq \mathcal{P}\omega$ is $\bigcup S$ and the approximation relation is $x \ll y \Leftrightarrow x \subseteq y \ \& \ x$ is finite.

1.2.3 The Partial Functions on the Naturals

Definition 1.2.1 A *partial function* $f : X \to Y$ between sets X and Y is a function $f : A \to Y$ defined on a subset $A \subseteq X$. We write $\mathrm{dom}\, f = A$ for the *domain* of a partial map $f : X \to Y$.

The set of partial mappings from $\mathbb{N}$ to $\mathbb{N}$

$$[\mathbb{N} \to \mathbb{N}] = \{ f \mid f : \mathbb{N} \to \mathbb{N} \textit{ is a partial map} \}$$

becomes an ω-algebraic dcpo when ordered by extension

$$f \sqsubseteq g \Leftrightarrow \mathrm{dom}\, f \subseteq \mathrm{dom}\, g \;\&\; f = g \text{ on } \mathrm{dom}\, f.$$

The supremum of a directed set $S \subseteq [\mathbb{N} \to \mathbb{N}]$ is $\bigcup S$, under the view that functions are certain subsets of $\mathbb{N} \times \mathbb{N}$, while the approximation relation is

$$f \ll g \Leftrightarrow f \sqsubseteq g \;\&\; \mathrm{dom}\, f \text{ is finite.}$$

1.2.4 The Cantor Set Model

The collection of functions

$$\Sigma^{\infty} = \{ s \mid s : \{1, \ldots, n\} \to \{0, 1\}, 0 \leq n \leq \infty \}$$

is also an ω-algebraic dcpo under the extension order

$$s \sqsubseteq t \Leftrightarrow |s| \leq |t| \;\&\; (\,\forall\, i \leq |s|\,)\; s(i) = t(i),$$

where $|s|$ is written for the cardinality of $\mathrm{dom}\, s$. The supremum of a directed set $S \subseteq \Sigma^{\infty}$ is $\bigcup S$, while the approximation relation is

$$s \ll t \Leftrightarrow s \sqsubseteq t \;\&\; |s| < \infty.$$

The extension order in this special case is usually called the *prefix* order. The elements $s \in \Sigma^{\infty}$ are called *strings* over $\{0, 1\}$. The quantity $|s|$ is called the *length* of a string s. The *empty string* ε is the unique string with length zero. It is the least element $\perp$ of Σ^{∞}.

1.2.5 The Domain of Lists

Let S be a set which carries an intrinsic partial order $\leq$.

Definition 1.2.2 A *list* over S is a function $x : \{1, ..., n\} \rightarrow S$, for $n \geq 0$. The *length* of a list x is $|\mathrm{dom}\, x|$. The set of all (finite) lists over S is $[S]$. A list x is *sorted* if x is monotone as a map between posets.

A list x can be written as $[x(1), ..., x(n)]$, where the *empty list* (the list of length 0) is written $[\,]$. We also write lists as $a :: x$, where $a \in S$ is the *first element* of the list $a :: x$, and $x \in [S]$ is the *rest* of the list $a :: x$. For example, the list $[1, 2, 3]$ is written $1 :: [2, 3]$.

Definition 1.2.3 A set $K \subseteq \mathbb{N}$ is *convex* if $a, b \in K$ & $a \leq x \leq b \Rightarrow x \in K$. Given a finite set $K \subseteq \mathbb{N}$, the map

$$\mathrm{scale}(K) : \{1, ..., |K|\} \rightarrow K$$

$$\mathrm{scale}(K)(i) = \min K + i - 1$$

relabels the elements of K so that they begin with one.

Definition 1.2.4 For $x, y \in [S]$, x is a *sublist* of y iff there is a convex subset $K \subseteq \{1, \ldots, \text{length } y\}$ such that $y \circ \mathrm{scale}\, K = x$.

Example 1.2.1 If $L = [1, 2, 3, 4, 5, 6]$, then $[1, 2, 3], [4, 5, 6], [3, 4, 5], [2, 3, 4], [3, 4], [5]$ and $[\,]$ are all sublists of L. However, $[1, 4, 5, 6], [1, 3]$ and $[2, 4]$ are *not* sublists of L.

We will see an easy way to prove the following in Section 2.2.

Lemma 1.2.1 *The finite lists $[S]$ over a set S, ordered by*

$$x \sqsubseteq y \Leftrightarrow y \textit{ is a sublist of } x,$$

form an algebraic dcpo with $[S] = K([S])$. Thus, $[S]$ is ω-continuous iff S is countable.

Often lists are ordered by either the *prefix* order or the *postfix* order. However, this is not adequate for describing algorithms which manipulate lists.

Consider the binary search of a sorted list L for a key k. To model the behavior of this algorithm, we must have an order which allows for the fact that sometimes binary search will check the *left* sublist of L for k, while other times it will search the *right* sublist of L. Using the *prefix* order, we cannot model the case when it searches the right sublist; using the *postfix* order, we cannot model the searching of the left sublist. Other basic examples show that the full sublist order is required.

Given that one must then use the sublist order, there is still the question of why we use the *reverse* sublist order. This will be clear in Chapter 4.

1.2.6 The Upper Space

If X is a locally compact Hausdorff space, its upper space

$$\mathbf{U}X = \{\emptyset \neq K \subseteq X : K \textit{ is compact}\}$$

ordered under reverse inclusion

$$A \sqsubseteq B \Leftrightarrow B \subseteq A$$

is a continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{U}X$ is $\bigcap S$ and the approximation relation is $A \ll B \Leftrightarrow B \subseteq \text{int}(A)$.

1.2.7 The Formal Ball Model

Given a metric space (X, d), the formal ball model [7]

$$\mathbf{B}X = X \times [0, \infty)$$

is a poset when ordered via

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leq r - s.$$

The approximation relation is characterized by

$$(x, r) \ll (y, s) \Leftrightarrow d(x, y) < r - s.$$

The poset $\mathbf{B}X$ is continuous. However, $\mathbf{B}X$ is a dcpo iff the metric d is complete. Finally, $\mathbf{B}X$ has a countable basis iff X is a separable metric space.

1.3 Computational Intuitions Underlying Domain Theory

In this section we want to discuss some basic philosophy, ideas and phrases we are likely to use in the course of this work that we wish to be reasonably clear about.

1.3.1 The Information Order

The phrase "information order" refers to the idea that the objects of a domain are being interpreted as information. In this way, the partial order $\sqsubseteq$ then becomes an ordering of information:

$$x \sqsubseteq y \Rightarrow y \text{ is at least as informative as } x.$$

A *maximal* element can then be viewed as being *ideal* information, while a nonmaximal is usually termed *partial* information. Ideal refers to something we usually approximate to arbitrary high levels of accuracy; For example, the integral of some function with an unknown antiderivative using Simpson's rule. Partial information, on the other hand, usually refers to things we can represent in perfect form on a machine, such as an integer, a truth value, or a character from the alphabet.

On the interval domain $\mathbf{I}\mathbb{R}$, consider the problem of computing the unique zero of

$$f(x) = x^3 - x + 1.$$

Let us call this zero r. According to the order on $\mathbf{I}\mathbb{R}$, $[r-1, r+1] \sqsubseteq [r]$, so $[r]$ ought to be more informative than $[r-1, r+1]$. This of course is true. In general, we can see that if $x \sqsubseteq y$, then y is at least as informative as x, by examining the lengths of the intervals. What is interesting here, however, is that the converse need not hold.

The interval $[r-3/2, r+1/4]$ is more informative than $[r-1, r+1]$ since it has smaller length. However, as members of $\mathbf{I}\mathbb{R}$, they are incomparable. Thus, this idea of an information order is not an equivalence, merely an implication. Essentially, this accounts for the myth that everything of interest on a domain is monotone.

1.3.2 Finite Approximation

Any object regarded a finite approximation must have a *representation* which is finite in nature that may be manipulated by finitary means. For example, a list with 51 integers qualifies. However, there are also infinite objects which work equally well: Though there is nothing finite about a nontrivial interval with rational endpoints, it may be represented and manipulated in a finite manner as a pair of rationals. Objects with such finite structure we think of as forming the *basis* for a domain. As the interval example shows, such a domain need not be algebraic.

We compute an ideal object x by computing a sequence (x_n) of finite approximations whose *limit* is x. A few examples will help illustrate this. Any interval in $\mathbf{I}\mathbb{R}$ may be computed as the limit of a sequence of intervals with rational endpoints; Any function in $[\mathbb{N} \to \mathbb{N}]$ is the limit of a sequence of partial functions whose domains are finite; Any string in Σ^∞ is the limit of a sequence of finite strings.

1.3.3 Domains as Data Types

A data type is a set of values and a set of operations on those values. They provide a means for classifying the objects we compute with so that compilers can ensure programs make a minimal amount of sense. For us, a domain will either serve to approximate the members of an idealized data type which lies along the top (that is, as a collection of maximal elements), or it will be viewed as a single data type by itself. An example of the former is $\mathbf{I}\mathbb{R}$, while for the latter we can take $[S]$.

1.3.4 Programs as Functions Between Domains

Well this makes enough sense. If a data type is a domain, why can't a program be a function between domains? Things are far from being this simple. Ultimately, it is simply dishonest to say that a program *is* a function between domains. An algorithm is *more* than just a function between domains.

For example, mergesort, quicksort and bubble sort all define the same function on the domain of lists,

$$\text{sort} : [S] \to [S],$$

whose input is a list and whose output is the list sorted. However, they are all different algorithms.

An algorithm is primarily concerned with *how* we get from an input to an output. Little else really matters. This is the reason for complexity analysis: Some algorithms get from input to output in a better way than do others. We see from this that special attention must be given to *how* mappings between domains are constructed. This is something we will be concerned with. However, it seems honest enough to say that a function on a domain *represents* an algorithm, with representation coming in various degrees!

1.4 Notation and Conventions

Definition 1.4.1 A *splitting* on a poset P is a function $s : P \to P$ with $x \sqsubseteq sx$ for all $x \in P$.

Throughout, we adopt the following conventions:

- P is either a *poset* or a *continuous poset* but the letters D and E are reserved for *continuous dcpo's* only.

- The order dual of a poset P is written P^*. Its order is the one opposite to P:
$$x \sqsubseteq_{P^*} y \Leftrightarrow y \sqsubseteq_P x.$$
In particular, $[0, \infty)^*$ denotes the nonnegative reals in their opposite order. Observe then that a function $\mu : P \to [0, \infty)^*$ is monotone iff $x \sqsubseteq y \Rightarrow \mu x \geq \mu y$ where $\geq$ is the usual ordering on $[0, \infty)$.

- For a function $\mu : P \to [0, \infty)^*$, $\ker \mu = \{x \in P : \mu x = 0\}$. This is called the *kernel* of μ.

- The elements of any set X may be given the flat order: $x \sqsubseteq y \Leftrightarrow x = y$. We sometimes denote the poset which results by $X^\flat$.

- If $f : X \to X$ is a function on a set X, its set of fixed points is $\mathrm{fix}(f) = \{x \in X : f(x) = x\}$.

- $\mathbb{N} := \{0, 1, 2, \cdots\}$ is the set of nonnegative integers.

- $\mathbb{Q}$ is the set of rational numbers.

Chapter 2

Measurement

The first thing we have to do is develop the notion of "information content" for objects in a domain. By this we mean the assignment of an abstract value to each object, which allows for the distinction of comparable objects, and is based on a single defining characteristic that all members of the domain possess. For example, the domain of a partial function, or the length of an interval. Any mapping which assigns such a value to elements of a domain could easily be called a measurement, as in "a measure of information content." However, this term will be reserved for the case when information content is expressible as a nonnegative real number. This is an aspect that will be motivated both theoretically and practically, as strict adherence to either viewpoint is unsettling.

We will also address the issue of existence of measurements, by showing that they exist on any ω-continuous dcpo. Of course, such a result is only included to reassure the reader that the idea is sufficiently general. The real proof of its usefulness lies in particular cases, where it is not the fact that *a measurement* exists, but that the *obvious and useful choice* is in fact a measurement. This is fundamental to the theory: Measurements are easy to spot, and very little effort is required to verify that a given mapping is a measurement. Such a question gets right to the heart of the *applicability* of a suggested theory, and should always be a top priority.

2.1 Degree of Approximation

In Domain theory, one studies domains, objects which possess a notion of approximation and convergence, but have no *degree* of approximation. What good is a notion of approximation if we do not have a mechanism for measuring how close we are to the ideal?

Degree of approximation is an idea made necessary by the prototypical test for the termination of an iterative process. Here we compute, to within $\varepsilon > 0$ accuracy, an ideal element r, by calculating a sequence of finite approximations (x_n), according to the following algorithm:

```
n := 0;
repeat
        n := n + 1;
        compute x_n;
until μx_n < ε;
```

where $\mu x \geq 0$ is a number expressing the degree to which x approximates r. In particular, $\mu r = 0$.

Such a test should be *reliable,* in the sense that, the smaller μx_n is, the more accurately x_n approximates r.

Example 2.1.1 Consider the domain $D = \mathbb{N}^\flat \cup \{\infty\}$, where one first orders $\mathbb{N}$ flatly, and then adjoins a top element ∞. Define a map $\mu : D \to [0, \infty)^*$ by

$$\mu x = \begin{cases} 1/2^x & \text{if } x \in \mathbb{N}; \\ 0 & \text{if } x = \infty. \end{cases}$$

Setting $x_n = n$ for $n \geq 1$, we see that the algorithm above will give nonsensical results in the computation of $r = \infty$, irrespective of the accuracy $\varepsilon > 0$ imposed: No matter how small we require μx_n, we are no closer to ∞ than we began. That is, μ does not provide a reliable test for the termination of the iterative process (x_n).

The problem in Example 2.1.1 is not with the domain D. Instead, it lies with the manner in which we have tried to measure the *information content* of its objects. We have measured the elements $\mathbb{N}^\flat$ as being increasingly more informative, when in reality, they are all equally informative.

The correct way to measure $\mathbb{N}^\flat \cup \{\infty\}$ is with $\mu : \mathbb{N}^\flat \cup \{\infty\} \to [0, \infty)^*$, given by

$$\mu x = \begin{cases} 1 & \text{if } x \in \mathbb{N}; \\ 0 & \text{if } x = \infty. \end{cases}$$

Using this definition of μ, we only get close to ∞ in measure if we have actually computed it, which is what we expect of a domain whose nature is discrete.

Notice that the desire for a reliable termination test has forced us to ask about the correct way to measure the amount of information each object contains. Of course, Example 2.1.1 does little to elucidate the difference between a mapping that expresses information content, and one which is merely Scott continuous.

To gain real insight into the problem, consider the interval domain $\mathbf{I}\mathbb{R}$ and the mapping $\mu : \mathbf{I}\mathbb{R} \rightarrow [0, \infty)^*$ which assigns to each interval its length. In this case, there is a delicate mathematical reason that the test for termination is reliable. When one chooses $\varepsilon > 0$, they are implicitly fixing an interval $[r - \varepsilon, r + \varepsilon] \ll [r]$, and relying on the fact that

$$\mu x_n < \varepsilon \ \& \ x_n \sqsubseteq [r] \Rightarrow [r - \varepsilon, r + \varepsilon] \ll x_n.$$

It is *this implication* which guarantees that intervals containing r become increasingly more accurate as their measures get smaller.

In fact, even more is true: Given *any* $I \ll [r]$, there is an $\varepsilon > 0$ such that $x \sqsubseteq [r]$ and $\mu x < \varepsilon$ together imply $I \ll x$.

In the case of a poset P and a monotone map $\mu : P \rightarrow [0, \infty)^*$, we can see this implication amounts to saying that for all Scott open sets $U \subseteq P$,

$$r \in U \Rightarrow (\ \exists\, \varepsilon > 0\ )\ \mu_\varepsilon(r) \subseteq U,$$

where $\mu_\varepsilon(r) = \{x \in P : x \sqsubseteq r \ \& \ \mu x < \varepsilon\}$ and $\mu r = 0$. As is easily seen, this condition also explains the difference between the two maps considered on $\mathbb{N}^\flat \cup \{\infty\}$.

We will take this condition as being the formalization of when a mapping μ provides a reliable termination test for an iterative process on P which computes r. Essentially, the axiom on which the present work rests is the following: Any mapping which provides a reliable termination test for all of the iterative processes (x_n) capable of computing r must *as a consequence* measure the information content of the objects (x_n) relative to r.

2.2 Information Content

We begin with a poset P and a domain E.

Definition 2.2.1 If $\mu : P \rightarrow E$ is a monotone map between a poset and a domain, we set

$$\mu_\varepsilon(x) := \{\ y \in P : y \sqsubseteq x \ \& \ \varepsilon \ll \mu y\ \},$$

for all $x \in P$ and $\varepsilon \in E$.

Observe that $x \in \mu_\varepsilon(x) \Leftrightarrow \mu_\varepsilon(x) \neq \emptyset$, and $y \in \mu_\varepsilon(x) \Rightarrow y \in \mu_\varepsilon(y) \subseteq \mu_\varepsilon(x)$, for all $x, y \in P$ and $\varepsilon \in E$.

Lemma 2.2.1 *For a monotone map $\mu : P \to E$ and any subset $X \subseteq P$,*

$$\{ \ \mu_\varepsilon(x) : x \in \downarrow\!X, \varepsilon \in E \ \}$$

is a basis for a topology on $\downarrow\!X$ which we will denote by μ_{X}.

proof This lemma holds even if E is only a continuous poset. First let $x \in \downarrow\!X$. By the continuity of E, $\exists\, \varepsilon \ll \mu x$. Then $x \in \mu_\varepsilon(x) \subseteq \downarrow\!X$. This shows that every point of the space is contained in some $\mu_\varepsilon(x)$. Second, suppose $z \in \mu_\varepsilon(x) \cap \mu_\delta(y)$. Then $\varepsilon, \delta \ll \mu z$. Using the directedness of $\twoheaddownarrow\!\mu z$, $\exists\, \alpha \ll \mu z$ with $\varepsilon, \delta \sqsubseteq \alpha$. This gives $z \in \mu_\alpha(z) \subseteq \mu_\varepsilon(x) \cap \mu_\delta(y)$, which proves the claim. $\Box$

Definition 2.2.2 A monotone map $\mu : P \to E$ from a poset to a domain *induces the Scott topology near $X \subseteq P$* provided that

$$(\ \forall\, U \in \sigma_P\ )(\ \exists\, V \in \mu_X\ )(\ U \cap X \subseteq V \subseteq U\ ).$$

We write this as $\mu \to \sigma_X$. We say μ *induces the Scott topology everywhere* if $\mu \to \sigma_P$.

We think of the set X as containing objects we'd like to compute with iterative processes on P. In accordance with the first section, $\mu \to \sigma_X$ means the mapping μ provides a reliable termination test for these iterative processes.

Lemma 2.2.2 *For a monotone map $\mu : P \to E$ and a subset $X \subseteq P$, the following are equivalent:*

(i) *The mapping μ induces the Scott topology near X.*

(ii) $(\forall\, U \in \sigma_P)(\forall x \in X)\, x \in U \Rightarrow (\ \exists\, \varepsilon \in E\ )\, x \in \mu_\varepsilon(x) \subseteq U$.

proof (i) $\Rightarrow$ (ii): Let U be a Scott open set around $x \in X$. Then, by (i), we know that $(\exists\, V \in \mu_X)\ U \cap X \subseteq V \subseteq U$. Since $x \in V$, $\exists\, y \in \downarrow\!X$ and $\varepsilon \in E$ such that $x \in \mu_\varepsilon(y) \subseteq V$. Then $x \in \mu_\varepsilon(x) \subseteq \mu_\varepsilon(y)$. (ii) $\Rightarrow$ (i): Again let U be a Scott open set. Using (ii), $\forall x \in U \cap X$, $(\exists\ \mu_{\varepsilon_x}(x))\ x \in \mu_{\varepsilon_x}(x) \subseteq U$. Setting $V := \bigcup \mu_{\varepsilon_x}(x)$ finishes the proof. $\Box$

The case $E = [0, \infty)^*$ is of particular interest to us.

Corollary 2.2.1 *If $\mu : P \to [0,\infty)^*$ is a monotone map and $X \subseteq P$, the following are equivalent:*

(i) *The mapping μ induces the Scott topology near X.*

(ii) *For all Scott open sets $U \subseteq P$ and for all $x \in X$, if $x \in U$, then*

$$(\exists\, \delta > 0\,)\; a \sqsubseteq x \;\&\; |\mu x - \mu a| < \delta \Rightarrow a \in U.$$

proof The nonnegative reals $[0,\infty)^*$ is a domain in its dual order

$$x \sqsubseteq y \Leftrightarrow x \geq y \;\&\; x \ll y \Leftrightarrow x > y,$$

where the symbols $\geq$ and $>$ refer to the usual orders on $[0,\infty)$. Then for any $x \in P$ and $\varepsilon \geq 0$, we see that

$$\mu_\varepsilon(x) = \{y \in P : y \sqsubseteq x \;\&\; \varepsilon \ll \mu y\} = \{y \in P : y \sqsubseteq x \;\&\; \mu y < \varepsilon\}.$$

For (i) $\Rightarrow$ (ii), set $\delta = \varepsilon - \mu x > 0$. For (ii) $\Rightarrow$ (i), set $\varepsilon = \mu x + \delta > 0$ $\square$

Corollary 2.2.1 says that μ induces the Scott topology near X if, whenever a point $x \in X$ lies in a Scott open set U, all points sufficiently close to x lie in U, where "sufficiently close" is made precise by the map μ. Hence, the Scott open sets in this case can be compared to open sets in a metric space.

Example 2.2.1 The collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{\, [a,b] : a,b \in \mathbb{R} \;\&\; a \leq b \,\}$$

ordered under reverse inclusion is a poset and the length function

$$\mu : \mathbf{I}\mathbb{R} \to [0,\infty)^*$$
$$\mu[a,b] = b - a$$

is monotone. The mapping μ induces the Scott topology everywhere on $\mathbf{I}\mathbb{R}$. Let $U \subseteq \mathbf{I}\mathbb{R}$ be a Scott open set around $[a,b]$. It is easy to see that

$$\bigsqcup_{n \geq 1} [a - 1/n, b + 1/n] = [a,b]$$

and since U is Scott open, $[a - 1/n, b + 1/n] \in U$, for some n. Finally,

$$\begin{aligned} \bar{x} \sqsubseteq [a,b] \;\&\; |\mu\bar{x} - \mu[a,b]| < 1/n \;&\Rightarrow\; [a - 1/n, b + 1/n] \sqsubseteq \bar{x} \\ &\Rightarrow\; \bar{x} \in U, \end{aligned}$$

which finishes the proof. Note that $\ker \mu = \{[x] : x \in \mathbb{R}\} = \max \mathbf{I}\mathbb{R}$.

However, information content is not always most naturally expressed as a real number.

Example 2.2.2 The set of partial functions

$$[\mathbb{N} \to \mathbb{N}] = \{ f \mid f : \mathbb{N} \to \mathbb{N} \textit{ is a partial function} \}$$

when ordered by extension

$$f \sqsubseteq g \Leftrightarrow \mathrm{dom}\, f \subseteq \mathrm{dom}\, g \;\&\; f = g \text{ on } \mathrm{dom}\, f$$

becomes a poset, where dom denotes the monotone function

$$\mathrm{dom} : [\mathbb{N} \to \mathbb{N}] \to \mathcal{P}\omega$$

$$\mathrm{dom}\, f = \{ x \in \mathbb{N} : f \textit{ is defined at } x \}.$$

The mapping dom induces the Scott topology everywhere. Let $U \subseteq [\mathbb{N} \to \mathbb{N}]$ be a Scott open set around $f \in [\mathbb{N} \to \mathbb{N}]$. The set $\mathrm{dom}\, f$ is countable, so we can write

$$\mathrm{dom}\, f = \bigcup X_n$$

where (X_n) is an increasing sequence of finite sets. Now let f_n be the restriction of f to X_n. It is clear that

$$\bigsqcup f_n = f,$$

and since U is Scott open, $f_n \in U$, for some n. Let $\varepsilon = \mathrm{dom}\, f_n$. Then

$$f \in \mathrm{dom}_\varepsilon(f) = \{g : g \sqsubseteq f \;\&\; \varepsilon \ll \mathrm{dom}\, g\},$$

since ε is a finite set (so compact in $\mathcal{P}\omega$), and

$$\begin{aligned} g \in \mathrm{dom}_\varepsilon(f) &\Rightarrow \varepsilon \ll \mathrm{dom}\, g \;\&\; g \sqsubseteq f \\ &\Rightarrow f_n \sqsubseteq g \\ &\Rightarrow g \in U, \end{aligned}$$

which proves that $\mathrm{dom} \to \sigma_{[\mathbb{N}\to\mathbb{N}]}$. Finally, just like the previous example, we have $\mathrm{dom}\, f \in \max \mathcal{P}\omega$ iff $f \in \max\, [\mathbb{N} \to \mathbb{N}]$.

Because of examples like the last one, it is useful to know that such maps compose.

Lemma 2.2.3 (Composition) *Suppose that $\mu : P \to E$ and $\lambda : E \to F$ are monotone maps with $\mu \to \sigma_X$ and $\lambda \to \sigma_Y$. If $\mu(X) \subseteq Y$, then $\lambda\mu \to \sigma_X$.*

proof Let $U \subseteq P$ be a Scott open set around $x \in X$. Then there is an $\varepsilon \in E$ with $x \in \mu_\varepsilon(x) \subseteq U$. But $\mu x \in Y$ and $\mu x \in \Uparrow\varepsilon$, so

$$(\,\exists\, \delta \in F)\; \mu x \in \lambda_\delta(\mu x) \subseteq \Uparrow\varepsilon.$$

We claim that $x \in (\lambda\mu)_\delta(x) \subseteq U$. First, $\delta \ll \lambda(\mu x)$, so $x \in (\lambda\mu)_\delta(x)$. Next,

$$y \in (\lambda\mu)_\delta(x) \Rightarrow \mu y \sqsubseteq \mu x \;\&\; \delta \ll \lambda(\mu y) \Rightarrow \mu y \in \lambda_\delta(\mu x) \subseteq \Uparrow\varepsilon,$$

which gives $y \in \mu_\varepsilon(x) \subseteq U$. Thus, $x \in (\lambda\mu)_\delta(x) \subseteq U$. $\Box$

The following lemma asserts the *reflection of suprema.*

Lemma 2.2.4 *Suppose $\mu : P \to E$ is monotone and that $\mu \to \sigma_X$. If $S \subseteq P$ has a directed image $\mu(S)$ and there is an $x \in X$ such that*

(i) *x is an upper bound for S, and*

(ii) *$\mu x = \bigsqcup \mu(S)$,*

then $\bigsqcup S = x$.

proof Let u be an upper bound of S and consider a Scott open set U around x. Since $\mu \to \sigma_X$, $\exists\, \varepsilon \in E$ such that $x \in \mu_\varepsilon(x) \subseteq U$. Now

$$\varepsilon \ll \mu x = \bigsqcup \mu(S),$$

so $\exists\ s \in S$ with $\varepsilon \ll \mu s$. Then $s \in \mu_\varepsilon(x) \subseteq U$. Because $s \sqsubseteq u$ and U is an upper set, we have $u \in U$. Thus, every open set containing x also contains u, which shows that $x \sqsubseteq u$. Hence, $\bigsqcup S = x$. $\Box$

The next lemma yields the *strict monotonicity of μ on X.*

Lemma 2.2.5 *If $\mu : P \to E$ is monotone and $\mu \to \sigma_X$, then*

$$(\,\forall\, x \in P\,)(\,\forall\, y \in X\,)\; x \sqsubseteq y \;\&\; \mu x = \mu y \Rightarrow x = y.$$

proof Suppose $\mu x = \mu y$ and that U is a Scott open set around y. Then $\exists\, \varepsilon \in E$ such that $y \in \mu_\varepsilon(y) \subseteq U$. Since $x \sqsubseteq y$ and $\varepsilon \ll \mu x = \mu y$, $x \in \mu_\varepsilon(y) \subseteq U$. Then every Scott open set containing y also contains x so $y \sqsubseteq x$. $\Box$

Corollary 2.2.2 *If $\mu : P \to E$ is monotone and $\mu \to \sigma_P$, then*

$$(\,\forall x \in P\,)\; \mu x \in \max E \Rightarrow x \in \max P$$

proof Let $x \sqsubseteq y$ and $\mu x \in \max E$. By monotonicity of μ, $\mu x \sqsubseteq \mu y$, and by maximality of μx, $\mu x = \mu y$. But now the strict monotonicity of μ given in Lemma 2.2.5 yields $x = y$. $\Box$

Proposition 2.2.1 *Suppose that $\mu : Q \to E$ is continuous and $\mu \to \sigma_Q$. Then for a function $f : P \to Q$ between posets, the following are equivalent:*

(i) *f is Scott continuous.*

(ii) *f is monotone and μf is Scott continuous.*

proof (ii) $\Rightarrow$ (i): Let $S \subseteq P$ be a directed set which has a supremum. First of all, $x := f(\bigsqcup S)$ is an upper bound for the directed set $f(S)$, since f is monotone. Since μf is continuous,

$$\mu x = \mu f(\bigsqcup S) = \bigsqcup \mu(f(S)).$$

Since μ reflects suprema, we know not only that $\bigsqcup f(S)$ exists, but that this supremum is equal to $x = f(\bigsqcup S)$. $\Box$

The ideas presented thus far lead to a methodical approach for proving that a poset is a domain.

Theorem 2.2.1 *Suppose that $\mu : P \to [0, \infty)^*$ is monotone and also strictly monotone: $x \sqsubseteq y$ & $\mu x = \mu y \Rightarrow x = y$. If every increasing sequence in P has a supremum preserved by μ, then*

(i) *P is a dcpo,*

(ii) *μ is Scott continuous as a map between dcpo's,*

(iii) *Every directed subset $S \subseteq P$ contains an increasing sequence whose supremum is $\bigsqcup S$,*

(iv) *For all $x, y \in P$, $x \ll y$ iff for every increasing sequence (x_n) in P,*

$$y \sqsubseteq \bigsqcup x_n \Rightarrow (\exists n)\, x \sqsubseteq x_n.$$

(v) *For all $x \in P$, $\downarrow\!x$ is directed with supremum x iff it contains an increasing sequence with supremum x.*

proof Let S be a directed subset of P and set $u = \inf\{\mu s : s \in S\}$. Define an increasing sequence (a_n) in S according to the following rule: First, choose $a_1 \in S$ with $u \leq \mu a_1 < u + 1$; Next, given $a_n \in S$ with $u \leq \mu a_n < u + 1/n$, there is $b_{n+1} \in S$ with $u \leq \mu b_{n+1} < u + 1/(n+1)$, so by the directedness of S, choose $a_{n+1} \in S$ with $a_n, b_{n+1} \sqsubseteq a_{n+1}$. The sequence is increasing so it has a supremum $x = \bigsqcup a_n$. We claim that $x = \bigsqcup S$. Let $s \in S$. Now define a sequence (c_n) in S by choosing $c_1 \in S$ with $s, a_1 \sqsubseteq c_1$ and given $c_n \in S$, choose $c_{n+1} \in S$ with $a_{n+1}, c_n \sqsubseteq c_{n+1}$. Then (c_n) is increasing so let $c = \bigsqcup c_n$. Now c is an upper bound for (a_n), which gives $x \sqsubseteq c$, while $u \leq \mu c_n \leq \mu a_n < u + 1/n$ implies that

$$\mu x = \mu(\bigsqcup a_n) = \lim_{n\to\infty} \mu a_n = \lim_{n\to\infty} \mu c_n = \mu(\bigsqcup c_n) = \mu c,$$

using the assumption that μ preserves sups of increasing sequences. But μ is strictly monotone on P, so we have $x = c$. Thus, $s \sqsubseteq c_1 \sqsubseteq c = x$. This proves that x is an upper bound for S. However, any upper bound for S is an upper bound for (a_n), and so must be above x. Hence, $\bigsqcup S = x$. (iii) is now immediate. For (ii),

$$\mu(\bigsqcup S) = \mu(\bigsqcup a_n) = \lim_{n\to\infty} \mu a_n = u = \inf\{\mu s : s \in S\}.$$

The proof of (iv) follows from (iii). For (v), the direction ($\Rightarrow$) is easy using (iii). For the other direction of (v) one can use Proposition 2.2.4 of [1]. $\square$

This result holds if $[0,\infty)^*$ is replaced with any first countable domain E.

Example 2.2.3 If $\mu : P \to \mathbb{N}^*$ is strictly monotone on a poset P, then

(i) The map μ induces the Scott topology everywhere, and

(ii) P is an algebraic domain with $K(P) = P$.

For (i), if $U \subseteq P$ is a Scott open set around $x \in P$, then

$$a \sqsubseteq x \ \&\ \mu x \leq \mu a < \mu x + 1 \Rightarrow a = x,$$

which means $x \in \mu_{(\mu x+1)}(x) = \{x\} \subseteq U$. Thus, $\mu \to \sigma_P$. For (ii), μ maps into $\mathbb{N}^*$, so every increasing sequence of distinct elements in P is finite. Now Theorem 2.2.1 applies.

Domains like this arise when one orders the members of a recursively defined data type according to substructures. For example, on the domain of lists $[S]$, the mapping

$$\text{len} : [S] \to \mathbb{N}^*$$

$$\text{len}\, L = \begin{cases} 0 & \text{if } L = [\,], \\ 1 + \text{len rest } L & \text{otherwise,} \end{cases}$$

where $\text{rest} : [S] \to [S]$ is given by

$$\text{rest}\, L = \begin{cases} [\,] & \text{if } L = [\,], \\ x & \text{if } L = a :: x, \end{cases}$$

is strictly monotone so $\text{len} \to \sigma_{[S]}$. Once again, $\text{len}\, L = 0 \Leftrightarrow L \in \max [S]$.

Theorem 2.2.1 can actually be applied to most of the domains we will consider. It will be of use to us in Chapter 8.

2.3 Information Content and Approximation

We now assume that P is a continuous poset.

Lemma 2.3.1 *Let P be a continuous poset with basis B. If $\mu : P \to E$ is monotone and $X \subseteq P$, then the following are equivalent:*

(i) *The mapping μ induces the Scott topology near X.*

(ii) $(\,\forall\, b \in B\,)(\,\forall\, x \in X\,)\; x \in \twoheaduparrow b \Rightarrow (\,\exists\, \varepsilon \in E\,)\; x \in \mu_\varepsilon(x) \subseteq \twoheaduparrow b.$

proof (i) $\Rightarrow$ (ii) holds because the sets $\twoheaduparrow b$ are Scott open in a continuous poset. (ii) $\Rightarrow$ (i): $B \subseteq P$ is a basis iff $B \cap \twoheaddownarrow x$ contains a directed subset with supremum x, for all $x \in P$. Then if U is a Scott open set around $x \in X$, there is $b \in B \cap \twoheaddownarrow x$ such that $b \in U$. By (ii), there is $\varepsilon \in E$ with $x \in \mu_\varepsilon(x) \subseteq \twoheaduparrow b \subseteq U$. $\square$

The theory on continuous posets is fundamentally different from the one on posets. The next proposition reveals why.

Proposition 2.3.1 *If P is a continuous poset, then for an upper set U,*

$$U \in \sigma_P \Leftrightarrow (\,\forall\, x \in U\,)(\,\exists\, a \ll x\,)\; \twoheaduparrow a \cap \downarrow x \subseteq U.$$

In fact, $\{\, \uparrow(\twoheaduparrow x \cap \downarrow y) : x, y \in P\,\}$ is a basis for σ_P.

proof For the first part, the $(\Rightarrow)$ direction holds because $\bigsqcup \twoheaddownarrow x = x$ in a continuous poset, while $(\Leftarrow)$ holds because continuous posets have the interpolation property. For the second part, let V be any Scott open set and W be any lower set. We will prove that the upper set $\uparrow(V \cap W)$ is in fact Scott open. Let $S \subseteq P$ be a directed subset with $\bigsqcup S \in \uparrow(V \cap W)$. Then $\exists\, y \in V \cap W$ with $y \sqsubseteq \bigsqcup S$. Since V is open, $\exists\, z \ll y$ & $z \in V$. Then for some $s \in S, z \sqsubseteq s$. Since W is a lower set, $z \in V \cap W$, which proves that some $s \in S$ hits $\uparrow(V \cap W)$. This proves that we have a Scott open set. $\Box$

Corollary 2.3.1 *The identity map on a domain induces the Scott topology everywhere.*

Theorem 2.3.1 *If $\mu : P \to E$ is Scott continuous and $\mu \to \sigma_X$, then*

$$\{\ \uparrow\mu_\varepsilon(x) \cap X : x \in X,\ \varepsilon \in E\ \}$$

is a basis for the Scott topology on X.

proof The continuity of μ and Proposition 2.3.1 imply that

$$\uparrow\mu_\varepsilon(x) = \uparrow(\ \mu^{-1}(\twoheaduparrow\varepsilon) \cap \downarrow x\ )$$

is Scott open in P. In the other direction, if $U \subseteq P$ is a Scott open set around $x \in X$, then using $\mu \to \sigma_X$, we see

$$\begin{aligned} x \in U &\Rightarrow (\ \exists\, \varepsilon \in E\ )\, x \in \mu_\varepsilon(x) \subseteq U \\ &\Rightarrow x \in \uparrow\mu_\varepsilon(x) \subseteq U \\ &\Rightarrow x \in \uparrow\mu_\varepsilon(x) \cap X \subseteq U \cap X. \end{aligned}$$

Finally, only *now* is it clear that the sets in question must form a basis for a topology on X, which, by the remarks above, must be the Scott topology. $\Box$

Corollary 2.3.2 *If $\mu : P \to E$ is continuous and $\mu \to \sigma_X$, then*

(i) *For all $V \in \mu_X$, $\uparrow V$ is Scott open in P.*

(ii) *For all $x \in X$ and $y \in P$, $y \ll x \Leftrightarrow (\ \exists\, \varepsilon \in E\ )\, x \in \mu_\varepsilon(x) \subseteq \uparrow y$.*

proof (i) The upper set of a union is the union of the upper sets. V is the union of sets whose upper sets are Scott open. (ii) $(\Rightarrow)$: $\twoheaduparrow y$ is a Scott open set around x. $(\Leftarrow)$: Then $x \in \uparrow\mu_\varepsilon(x) \subseteq \uparrow y$ and $\uparrow\mu_\varepsilon(x)$ is Scott open. $\Box$

The next corollary confirms the fact that we do not approximate compact elements; Rather, we compute them exactly.

Corollary 2.3.3 *If $\mu : P \to E$ is continuous and $\mu \to \sigma_X$, then for $x \in X$, the following are equivalent:*

(i) *The element x is compact in P, that is, $x \in K(P)$.*

(ii) *There is an $\varepsilon \in E$ with $\mu_\varepsilon(x) = \{x\}$.*

(iii) *The set $\{x\}$ is open in the μ_X topology.*

proof (i) $\Rightarrow$ (ii): By the characterization of $\ll$, $(\exists\, \varepsilon \in E)\ x \in \mu_\varepsilon(x) \subseteq \uparrow\! x$. But then we must have $\mu_\varepsilon(x) = \{x\}$. (ii) $\Rightarrow$ (iii): $\mu_\varepsilon(x) = \{x\}$ is a basic open set in the μ_X topology. (iii) $\Rightarrow$ (i): The upper set of *any* μ_X-open set is Scott open. Then $\uparrow\! x$ is Scott open, i.e., x is compact in P. $\square$

Proposition 2.3.2 *If $\mu : P \to E$ is continuous and $\mu \to \sigma_X$, then $x \in X$ is the supremum of an increasing sequence of approximations in P provided the same is true of μx.*

proof Let $S \subseteq P$ be a directed set with $x = \bigsqcup S \in X$. We will find an increasing sequence in S whose supremum is x. First write $\mu x = \bigsqcup e_n$, where $e_n \ll \mu x$, and (e_n) is increasing. By the continuity of μ,

$$e_n \ll \mu x = \mu(\bigsqcup S) = \bigsqcup \{\, \mu s : s \in S \,\}.$$

Then $(\exists a_1 \in S) e_1 \sqsubseteq \mu a_1$. Given $a_n \in S$ with $e_n \sqsubseteq \mu a_n$, first choose $b_{n+1} \in S$ with $e_{n+1} \sqsubseteq \mu b_{n+1}$, and then by the directedness of S,

$$(\, \exists\, a_{n+1} \in S\,)\ a_n, b_{n+1} \sqsubseteq a_{n+1}.$$

Then (a_n) in S is increasing and satisfies $e_n \sqsubseteq \mu a_n$, for all n. But $a_n \sqsubseteq x$ for all n, so $\bigsqcup \mu a_n = \mu x$. By Lemma 2.2.4, $\bigsqcup a_n = x$. Finally, P is a continuous poset, so taking $S = \twoheaddownarrow x$ finishes the proof. $\square$

Proposition 2.3.3 *If $\mu : P \to E$ is continuous and $\mu \to \sigma_X$, then*

$$(\ \forall x \in X\ )\ \mu x \in K(E) \Rightarrow x \in K(P).$$

proof Suppose that $x = \bigsqcup S$, for a directed set $S \subseteq P$. Then $\mu x = \bigsqcup \mu(S)$. The compactness of μx means $(\exists s \in S)\ \mu x \sqsubseteq \mu s$. However, $s \sqsubseteq x$, so we must have $\mu x = \mu s$. By the strictness of μ, $s \sqsubseteq x \ \&\ \mu x = \mu s \Rightarrow x = s$. Consequently, $x \in S$. Since P is a continuous poset, taking $S = \twoheaddownarrow x$ proves that $x \ll x$, i.e, x is compact. $\square$

The results on continuous posets need not hold for posets in general.

Example 2.3.1 Let $P = \{a_n : n \geq 1\} \cup \{b_n : n \geq 1\} \cup \{\infty\}$ where (a_n) and (b_n) are distinct copies of $\mathbb{N}$ ordered so that the supremum of each is ∞. Define $\mu : P \to [0, \infty)^*$ by $\mu\infty = 0$, $\mu a_n = \mu b_n = 1/2^n$, $n \geq 1$. Then

(i) μ is Scott continuous and $\mu \to \sigma_P$,

(ii) P is a dcpo which is not continuous,

(ii) $\mu_1(a_1) = \{a_1\}$ but a_1 is not compact in P,

(iii) $\uparrow\mu_1(a_1) = \{a_n : n \geq 1\} \cup \{\infty\}$ is not Scott open in P,

(iv) $\mu a_1 = 1/2 \in K(\mathrm{Im}\,\mu)$ but $a_1 \notin K(P)$.

We prove $\mu \to \sigma_P$. If $U \subseteq P$ is Scott open and $\infty \in U$, then there is $K \geq 1$ with $a_n, b_n \in U$ for all $n \geq K$. Then $\infty \in \mu_{1/2^K}(\infty) \subseteq U$. The other cases are simple, because the elements a_n and b_n can be isolated with μ: For $n \geq 0$, $\mu_{1/2^n}(a_{n+1}) = \{a_{n+1}\}$ and $\mu_{1/2^n}(b_{n+1}) = \{b_{n+1}\}$.

On a continuous poset, sometimes *directedness* is also reflected.

Lemma 2.3.2 *Suppose $\mu : P \to E$ is monotone on a continuous poset P and that $\mu \to \sigma_X$. For any $x \in X$ and every $S \subseteq \mathord{\downarrow\!\!\downarrow} x$, if $\mu(S)$ is directed with supremum μx, then S is directed with supremum x.*

proof Lemma 2.2.4 implies that $\bigsqcup S = x$. For the directedness of S, let $a, b \in S$. Then $x \in \mathord{\uparrow\!\!\uparrow} a \cap \mathord{\uparrow\!\!\uparrow} b$. The set $\mathord{\uparrow\!\!\uparrow} a \cap \mathord{\uparrow\!\!\uparrow} b$ is Scott open because P is a continuous poset. Thus, there is $\varepsilon \in E$ with $x \in \mu_\varepsilon(x) \subseteq \mathord{\uparrow\!\!\uparrow} a \cap \mathord{\uparrow\!\!\uparrow} b$. Next, $\varepsilon \ll \mu x = \bigsqcup \mu(S)$, so there is $c \in S$ with $\varepsilon \ll \mu c$, using interpolation in E. But then $c \in \mu_\varepsilon(x) \subseteq \mathord{\uparrow\!\!\uparrow} a \cap \mathord{\uparrow\!\!\uparrow} b$. Hence, $a, b \sqsubseteq c$. $\Box$

The converse holds if μ is continuous and $E = [0, \infty)^*$.

Proposition 2.3.4 *If $\mu : P \to [0, \infty)^*$ is Scott continuous, the following are equivalent:*

(i) *The mapping μ induces the Scott topology near X.*

(ii) *For all $x \in X$, if (x_n) is a sequence in $\mathord{\downarrow\!\!\downarrow} x$ with $\lim \mu x_n = \mu x$, then (x_n) is directed with supremum x.*

proof (i) ⇒ (ii) Apply Lemma 2.3.2 with $E = [0, \infty)^*$. (ii) ⇒ (i) Let U be a Scott open set around $x \in X$. Let $\delta_n = \mu x + 1/n$ for $n \geq 1$. By way of contradiction, assume that $\mu_{\delta_n}(x) \not\subseteq U$, for all n. Then for each n, there is $x_n \sqsubseteq x$ with $\mu x \leq \mu x_n < \delta_n$ and $x_n \notin U$. By the continuity of μ, we can assume $x_n \ll x$. By (ii), (x_n) is directed with supremum x. But U is Scott open, so this implies that $x_n \in U$, for some n, a contradiction. ☐

Corollary 2.3.4 *For a function $\mu : P \to [0, \infty)^*$ on a continuous poset P, the following are equivalent:*

(i) *The map μ is Scott continuous and induces the Scott topology everywhere.*

(ii) *For all $x \in P$ and every $S \subseteq \mathord{\downarrow\!\!\downarrow} x$, S is directed with supremum x iff $\inf\{\mu s : s \in S\} = \mu x$.*

This allows us to further develop the technique of Theorem 2.2.1: We now have a method for detecting bases in continuous posets.

Proposition 2.3.5 *If P is a continuous poset and $\mu : P \to [0, \infty)^*$ is a Scott continuous map with $\mu \to \sigma_P$, then for $B \subseteq P$, the following are equivalent:*

(i) *B is a basis for P.*

(ii) *For all $x \in P$, $B \cap \mathord{\downarrow\!\!\downarrow} x$ contains a sequence (x_n) with $\lim \mu x_n = \mu x$.*

proof (i) ⇒ (ii) First, $B \cap \mathord{\downarrow\!\!\downarrow} x$ is directed with supremum x. The continuity of μ implies $\mu x = \inf\{\mu b : b \in B \cap \mathord{\downarrow\!\!\downarrow} x\}$. Now (ii) follows. (ii) ⇒ (i) If $B \cap \mathord{\downarrow\!\!\downarrow} x$ contains a sequence (x_n) with $\lim \mu x_n = \mu x$, then $\inf\{\mu b : b \in B \cap \mathord{\downarrow\!\!\downarrow} x\} = \mu x$. By Lemma 2.3.2 with $E = [0, \infty)^*$, $B \cap \mathord{\downarrow\!\!\downarrow} x$ is directed with supremum x. ☐

Example 2.3.2 Given a metric space (X, d), the formal ball model [7]

$$\mathbf{B}X = X \times [0, \infty)$$

ordered by

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leq r - s$$

is a poset whose approximation relation is $(x, r) \ll (y, s) \Leftrightarrow d(x, y) < r - s$.

$\mathbf{B}X$ is a continuous poset since $(x, r + 1/n) \in \mathord{\downarrow\!\!\downarrow}(x, r)$ defines an increasing sequence with supremum (x, r). One can show that $\mathbf{B}X$ is a dcpo iff the metric d is complete.

The natural projection

$$\pi : \mathbf{B}X \to [0,\infty)^*$$

$$\pi(x,r) = r$$

is easily seen to be Scott continuous. In fact, π induces the Scott topology everywhere on $\mathbf{B}X$. Let $U \subseteq \mathbf{B}X$ be a Scott open set around $(x,r) \in \mathbf{B}X$. As mentioned above,

$$\bigsqcup (x, r+1/n) = (x,r)$$

and since U is Scott open, $(x, r+1/n) \in U$, for some n. First, we note that $(x,r) \in \pi_{r+\frac{1}{2n}}(x,r)$. Next,

$$\begin{aligned} (y,s) \in \pi_{r+\frac{1}{2n}}(x,r) &\Rightarrow (y,s) \sqsubseteq (x,r) \;\&\; \pi(y,s) = s < r + 1/2n \\ &\Rightarrow d(x,y) \leq s - r < r - s + 1/n \\ &\Rightarrow (x, r+1/n) \sqsubseteq (y,s) \\ &\Rightarrow (y,s) \in \uparrow\!(x, r+1/n) \subseteq U, \end{aligned}$$

which means $(x,r) \in \pi_{r+\frac{1}{2n}}(x,r) \subseteq U$. This proves $\pi \to \sigma_{\mathbf{B}X}$.

If A is a dense subset of X, then $A \times \mathbb{Q}^+$ is a basis for $\mathbf{B}X$, where $\mathbb{Q}^+ = \mathbb{Q} \cap [0,\infty)$. Let $(x,r) \in \mathbf{B}X$ and choose a sequence of rationals $q_n > r$ with $q_n \to r$. By the density of A,

$$(\,\forall n\,)(\,\exists\, a_n \in A\,)\; d(a_n, x) < q_n - r.$$

Then $(a_n, q_n) \ll (x,r)$ and $\lim \pi(a_n, q_n) = \lim q_n = r$. By Proposition 2.3.5, $A \times \mathbb{Q}^+$ is a basis for $\mathbf{B}X$.

If $f : X \to Y$ is a Lipschitz map between metric spaces with Lipschitz constant k, then

$$\bar{f} : \mathbf{B}X \to \mathbf{B}Y$$

$$\bar{f}(x,r) = (fx, k \cdot r)$$

is monotone and $\pi\bar{f} = k \cdot \pi$ is Scott continuous. By Propsition 2.2.1, the extension $\bar{f}$ is Scott continuous.

Once again, we see that $\ker \pi = \{(x,0) : x \in X\} = \max \mathbf{B}X$.

Example 2.3.1 shows that a poset with a mapping $\mu \to \sigma_P$ need not be continuous. By the last example, a continuous poset with a map $\mu \to \sigma_P$ need not be a domain. There is one final example we should mention now.

Example 2.3.3 On the Cantor set model Σ^∞, the Scott continuous map

$$|\cdot| : \Sigma^\infty \to \mathbb{N} \cup \{\infty\},$$

which takes a string to its length, induces the Scott topology everywhere. To see this, suppose $s \ll x$ in Σ^∞. Then s is finite. Hence,

$$a \sqsubseteq x \;\&\; |a| \geq |s| \Rightarrow s \sqsubseteq a.$$

In this example, Corollary 2.2.2 provides a formal explanation for why

$$|s| = \infty \Rightarrow s \in \max \Sigma^\infty \Rightarrow s \text{ is infinite},$$

while Proposition 2.3.3 justifies

$$|s| = n \in \mathbb{N} \Rightarrow s \in K(\Sigma^\infty) \Rightarrow s \text{ is finite}.$$

Finally, if we compose $|\cdot|$ with the isomorphism

$$(\{0\} \cup \{1/2^n : n \geq 0\})^* \simeq \mathbb{N} \cup \{\infty\},$$

we obtain a map

$$\frac{1}{2^{|\cdot|}} : \Sigma^\infty \to [0,\infty)^*$$
$$s \mapsto \frac{1}{2^{|s|}}$$

that induces the Scott topology everywhere with $\ker 1/2^{|\cdot|} = \max \Sigma^\infty$. We think of $1/2^{|s|}$ as measuring the probability of observing s.

We have seen that a mapping $\mu : P \to E$ with $\mu \to \sigma_X$ tends to reflect the nature of E back onto X. Put another way, if we *measure* an element x as having property P, then it *does have* property P. For example, an upper bound measured as the supremum of a set S *is its supremum* (Lemma 2.2.4); comparable elements measured as equal *are equal* (Lemma 2.2.5).

2.4 Measurement

Definition 2.4.1 A *measurement* on a continuous poset P is a Scott continuous map $\mu : P \to [0,\infty)^*$ which induces the Scott topology near $\ker \mu$.

We usually write a poset P and its natural measurement μ as a pair (P, μ).

Example 2.4.1 From the previous two sections,

(i) $(\mathbf{I}\mathbb{R}, \mu)$ the interval domain,

(ii) $(\Sigma^\infty, 1/2^{|\cdot|})$ the Cantor set model,

(iii) $(\mathbf{B}X, \pi)$ the formal ball model, and

(iv) $([S], \mathrm{len})$ the domain of lists over S,

are all examples of continuous posets P with measurements μ that induce the Scott topology everywhere. In each case, we also have $\ker \mu = \max P$.

However, some measurements yield the Scott topology *only at the top.*

Example 2.4.2 Let (X, d) be a locally compact metric space. Its upper space

$$\mathbf{U}X = \{\emptyset \neq K \subseteq X : K \textit{ is compact}\}$$

is a continuous dcpo and the diameter mapping

$$\lambda : \mathbf{U}X \to [0, \infty)^*$$

$$\lambda K = \sup\{d(x, y) : x, y \in K\}$$

is a measurement with $\ker \lambda = \{\{x\} : x \in X\} = \max \mathbf{U}X$.

First, λ is Scott continuous. Let $\{K_i : i \in I\}$ be a filtered collection of compact subsets of X indexed by the set I. I is a poset in the natural way: $i \leq j \Leftrightarrow K_j \subseteq K_i$. Now choose a fixed set K_n. By the compactness of each K_i,

$$(\forall i \geq n)(\exists a_i, b_i \in K_i)\; \lambda K_i = d(a_i, b_i).$$

Since $a_i, b_i \in K_n$ for all $i \geq n$ and K_n is compact, each net (a_i) and (b_i) has a convergent subnet. Without loss of generality, we can assume there are points $a, b \in K_n$ with $a_i \to a$ & $b_i \to b$. The collection $\{K_i : i \in I\}$ is filtered so $a, b \in \bigcap_{i \geq n} K_i$. Then $d(a, b) \leq \lambda K_i = d(a_i, b_i)$, for all $i \geq n$, while the continuity of d gives $\lim_{i \geq n} d(a_i, b_i) = d(a, b)$. Together these imply that $\inf_{i \geq n} d(a_i, b_i) = d(a, b)$. Thus,

$$\inf_{i \in I} \lambda K_i \leq \inf_{i \geq n} \lambda K_i = \inf_{i \geq n} d(a_i, b_i) = d(a, b) \leq \lambda(\bigcap K_i),$$

while the monotonicity of λ yields $\lambda(\bigcap K_i) \leq \inf \lambda K_i$. This establishes the continuity of λ. To calculate its kernel, just note that

$$K \in \ker \lambda \Leftrightarrow (\exists x \in X)\; K = \{x\} \Leftrightarrow K \in \max \mathbf{U}X.$$

Finally, λ is a measurement. Suppose $K \ll \{x\} \in \ker \lambda$. Then $x \in \text{int}(K)$ so there is $\varepsilon > 0$ with $x \in B_\varepsilon(x) \subseteq \text{int}(K)$. If $L \sqsubseteq \{x\}$ with $\lambda L < \varepsilon$, then

$$y \in L \Rightarrow d(x,y) \leq \lambda L < \varepsilon \Rightarrow y \in B_\varepsilon(x) \subseteq \text{int}(K),$$

which means $L \subseteq \text{int}(K)$, that is, $K \ll L$. Hence, λ is a measurement. However, it need not induce the Scott topology everywhere: It may fail to be strictly monotone. For example, if $X = \mathbb{R}$ is the real line in its usual metric, we see that $[0,1] \sqsubseteq \{0,1\}$ & $\lambda[0,1] = 1 = \lambda\{0,1\}$ while $[0,1] \neq \{0,1\}$.

We now consider a few characteristics that all measurements possess.

Lemma 2.4.1 *If* $\mu : P \to [0,\infty)^*$ *is a measurement, then*

(i) $\ker \mu \subseteq \max P$.

(ii) $\ker \mu$ *is a* G_δ *in* P.

proof (i) Let $x \sqsubseteq y$ with $x \in \ker \mu$. By monotonicity, $y \in \ker \mu$. Then by strictness of μ on $\ker \mu$, $x = y$. (ii) $\ker \mu = \bigcap \mu^{-1}[0, 1/n)$. Since μ is continuous, this is the intersection of countably many Scott open sets. □

Since the smaller the measure, the greater an object is in information content, the first point in the last result is important: All objects with measure zero are maximal in the information order.

Proposition 2.4.1 *Suppose* $\mu : P \to [0,\infty)^*$ *is monotone and* $\mu \to \sigma_X$. *If* $x \in X$ *and* $S \subseteq {\downarrow} x$ *is nonempty with* $\inf\{\mu a : a \in S\} = \mu x$, *then* $\bigsqcup S = x$. *In addition, if* $S \subseteq \twoheaddownarrow x$, S *is directed.*

proof Use Lemmas 2.2.4 and 2.3.2 with $E = [0,\infty)^*$. □

The properties mentioned above arise frequently in applications.

Example 2.4.3 A function $f : \mathbb{R} \to \mathbb{R}$ is *unimodal* on an interval $[a,b]$ if it has a maximum value assumed at a unique point $x^* \in [a,b]$ such that

(i) f is strictly increasing on $[a, x^*]$, and

(ii) f is strictly decreasing on $[x^*, b]$.

Unimodal functions have the important property that

$$x_1 < x_2 \Rightarrow \begin{cases} x_1 \leq x^* \leq b & \text{if } f(x_1) < f(x_2), \\ a \leq x^* \leq x_2 & \text{otherwise.} \end{cases}$$

This observation leads to an algorithm for computing x^*. For a function $f : \mathbb{R} \to \mathbb{R}$ and a constant $1/2 < r < 1$, define

$$\max_f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$$

by

$$\max_f[a, b] = \begin{cases} [l(a, b), b] & \text{if } f(l(a, b)) < f(r(a, b)), \\ [a, r(a, b)] & \text{otherwise.} \end{cases}$$

where $l(a, b) = (b - a)(1 - r) + a$ and $r(a, b) = (b - a)r + a$. The mapping $\max_f$ is well-defined since

$$a < b \Rightarrow a \leq l(a, b) < r(a, b) \leq b.$$

If $f : \mathbb{R} \to \mathbb{R}$ is unimodal on $[a, b]$, then

$$\bigsqcup_{n \geq 0} \max_f^n[a, b] = [x^*].$$

For the proof, we first note that

$$\begin{aligned} x^* \in [a, b] \quad &\Rightarrow \quad a \leq l(a, b) < r(a, b) \leq b \text{ or } a = b \\ &\Rightarrow \quad \begin{cases} l(a, b) \leq x^* \leq b & \text{if } f(l(a, b)) < f(r(a, b)), \\ a \leq x^* \leq r(a, b) & \text{otherwise.} \end{cases} \\ &\Rightarrow \quad x^* \in \max_f[a, b], \end{aligned}$$

and so by induction, $\max_f^n[a, b] \sqsubseteq [x^*]$, for all $n \geq 0$. But $\mu \max_f x = r \cdot \mu x$, for all $x \in \mathbf{I}\mathbb{R}$, where μ is the length measurement, so

$$\lim_{n \to \infty} \mu \max_f^n[a, b] = \lim_{n \to \infty} r^n \mu[a, b] = 0 = \mu[x^*].$$

By Proposition 2.4.1, $\bigsqcup \max_f^n[a, b] = [x^*]$.

Finally, observe that $\max_f$ is not monotone. Let -1 $< \alpha < 1$ and $f(x) = 1 - x^2$. The function f is unimodal on any compact interval. Since $\max_f[\text{-}1, 1] = [\text{-}1, 2r - 1]$, we see that

$$\begin{aligned} \max_f[\text{-}1, 1] \sqsubseteq \max_f[\alpha, 1] \quad &\Rightarrow \quad 1 \leq 2r - 1 \text{ or } r(\alpha, 1) \leq 2r - 1 \\ &\Rightarrow \quad 1 \leq r \text{ or } \alpha + 1 \leq r(\alpha + 1) \\ &\Rightarrow \quad r \geq 1, \end{aligned}$$

which contradicts $r < 1$. Thus, for no value of r is the algorithm monotone.

In the last example, the function $\max_f$ has the entire real line as its set of fixed points, but only one of these has any computational significance: $[x^*]$. The measurement μ provides us with a mechanism for distinguishing this fixed point from all others, in a situation where the traditional methods of domain theory cannot be employed.

This illustrates the computational nature of the measurement property cited in Proposition 2.4.1. In fact, it plays a fundamental role in their characterization.

Theorem 2.4.1 (The internal characterization) *A continuous poset P admits a measurement $\mu : P \to [0,\infty)^*$ with $\ker\mu = X$ if and only if*

(i) *$X = \bigcap U_n$ is the intersection of countably many Scott open sets, and*

(ii) *For every $x \in X$, if (x_n) is a sequence with $x_n \ll x$ and $x_n \in U_n$, for all n, then (x_n) is directed with supremum x.*

proof For $(\Rightarrow)$, take $U_n = \mu^{-1}[0, 1/2^n)$ for $n \geq 0$. In the other direction, write $X = \bigcap_{n\geq 1} U_n$ as the intersection of a descending family of Scott open sets and note that (i) and (ii) still hold. With $U_0 = P$, define a mapping

$$n : P \to \mathbb{N} \cup \{\infty\}$$

$$n(x) = \sup\{n : x \in U_n, n \geq 0\}.$$

Observe that $n(x) = \infty$ iff $x \in X$. Next, define $\mu : P \to [0,\infty)^*$ by

$$\mu x = \frac{1}{2^{n(x)}}.$$

First, for $n \geq 0$, we have (a) $\mu^{-1}[0, 1/2^n] = U_n$ and (b) $\mu^{-1}[0, 1/2^n) = U_{n+1}$. For (a), $x \in U_n \Leftrightarrow n(x) \geq n \Leftrightarrow \mu x = 1/2^{n(x)} \leq 1/2^n$, and similarly for (b), $x \in U_{n+1} \Leftrightarrow n(x) \geq n+1 \Leftrightarrow n(x) > n \Leftrightarrow \mu x < 1/2^n$. To see that μ is Scott continuous, given $\varepsilon > 0$, choose the least $n \geq 0$ with $1/2^n < \varepsilon$. Then

$$\mu^{-1}[0, \varepsilon) = \mu^{-1}[0, 1/2^n] = U_n.$$

For the kernel, just note that $\ker\mu = \{x \in P : n(x) = \infty\} = X$. Finally, to prove that $\mu \to \sigma_X$, let $U \subseteq P$ be a Scott open set around $x \in X$. Suppose by way of contradiction that $(\forall\varepsilon > 0)\ \mu_\varepsilon(x) \not\subseteq U$. Then

$$(\forall n \geq 0)(\exists\, b_n \sqsubseteq x)\ \mu b_n < 1/2^n \ \&\ b_n \notin U.$$

Since $b_n \in \mu^{-1}[0, 1/2^n) = U_{n+1}$, we can choose $x_n \in U_{n+1}$ with $x_n \ll b_n$. Observe that $x_n \notin U$ since U is a upper set and $b_n \notin U$. Next,

$$(\forall n \geq 0)\, x_n \ll x \;\&\; x_n \in U_{n+1},$$

and so (x_n) is directed with supremum x by (ii). But $P \setminus U$ is Scott closed and $x_n \in P \setminus U$, for all n, so $x = \bigsqcup x_n \in P \setminus U$, which gives the desired contradiction. $\square$

Then measurements are special kinds of G_δ subsets of the top.

Corollary 2.4.1 *A subset $X \subseteq P$ of a continuous poset P is a G_δ iff there exists a Scott continuous mapping $\mu : P \to [0, \infty)^*$ with $\ker \mu = X$. In addition, if $X = \bigcap_{n \geq 1} U_n$ is the intersection of a decreasing sequence of Scott open sets, μ can be chosen so that*

$$\mu x \leq \frac{1}{2^n} \Leftrightarrow x \in U_n.$$

proof The function defined in the internal characterization of measurement works here as well. $\square$

The last corollary yields a useful technique for constructing measurements.

Lemma 2.4.2 *If X is a G_δ subset of a continuous poset P with a measurement $\mu : P \to [0, \infty)^*$, then there is a measurement $\lambda : P \to [0, \infty)^*$ such that*

$$\mu \leq \lambda \;\&\; \ker \lambda = \ker \mu \cap \mathrm{X}.$$

proof By the last corollary, there is a Scott continuous map $\sigma : P \to [0, \infty)^*$ with $\ker \sigma = X$. Now define $\lambda : P \to [0, \infty)^*$ by

$$\lambda x = \mu x + \sigma x.$$

First, $\lambda x = 0 \Leftrightarrow \mu x = 0 \;\&\; \sigma x = 0 \Leftrightarrow x \in \ker \mu \cap X$. Next, $\mu x \leq \lambda x$, so $\lambda_\varepsilon(x) \subseteq \mu_\varepsilon(x)$ for $x \in \ker \lambda$. Thus, λ is a measurement since μ is. $\square$

Another corollary is that all measurements can be assumed to map into $(\{0\} \cup \{1/2^n : n \geq 0\})^* \simeq \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$. However, it is important to realize that doing so normally results in a serious loss of information.

Example 2.4.4 If $\mu : P \to \mathbb{N}^\infty$ is a Scott continuous map which induces the Scott topology everywhere, then

$$\begin{aligned} x \in P \setminus \max P &\Rightarrow \mu x \neq \infty \\ &\Rightarrow (\,\exists\, n \in \mathbb{N}\,)\, \mu x = n \\ &\Rightarrow \mu x \in K(\mathbb{N}^\infty) \\ &\Rightarrow x \in K(P), \end{aligned}$$

which means P is an algebraic poset where all elements are compact except possibly the maximal elements. Hence, if we assume a measurement maps into $(\{0\} \cup \{1/2^n : n \geq 0\})^*$, we normally lose the ability to assume $\mu \to \sigma_P$.

2.5 Existence

Theorem 2.5.1 *Every ω-continuous dcpo D has a measurement*

$$\mu : D \to [0, \infty)^*$$

such that $\mu \to \sigma_D$ and for which the following are equivalent:

(i) $\ker \mu = \max D$.

(ii) *The relative Scott and Lawson topologies on* $\max D$ *agree.*

proof The reader is advised to see [16] for a nice illustration of important techniques. The ω-continuous dcpo D embeds as the set of coprimes in the continuous lattice of its Scott closed sets $\Gamma(D)$. The Lawson closure of its image in $\Gamma(D)$ is an order compactification denoted by $F(D)$ — this is called the *Fell order compactification.* Since D is ω-continuous, $F(D)$ is a compact metrizable partially ordered space, and so it admits a metric $\bar{d}$ which is radially convex:

$$x \sqsubseteq y \sqsubseteq z \Rightarrow \bar{d}(x, z) = \bar{d}(x, y) + \bar{d}(y, z)$$

for $x, y, z \in F(D)$. The restriction of $\bar{d}$ to D we denote by d. Specifically,

$$d(x, y) = \bar{d}(\downarrow x, \downarrow y)$$

for $x, y \in D$. This metric is also radially convex and yields the Lawson topology on D. Now define $\mu : D \to [0, \infty)^*$ by

$$\mu x = \sup\{\ \bar{d}(\downarrow x, y) : x \in y \in F(D)\ \}$$

Note that the supremum exists since the space $F(D)$ is compact. Now, if $x \sqsubseteq y \in w \in F(D)$, then by radial convexity,

$$\bar{d}(\downarrow y, w) \leq \mu x - \bar{d}(\downarrow x, \downarrow y)$$

and so $\mu y \leq \mu x - d(x, y)$. This in particular proves monotonicity into $[0, \infty)^*$. The Scott continuity of μ was proven in [16], where it is called upper semicontinuity. Now let U be any Scott open set around $y \in D$. Since U is Lawson open, $\exists\, \varepsilon > 0$ with $y \in B_\varepsilon(y) \subseteq U$. Set $\delta := \varepsilon + \mu y$. Clearly, $y \in \mu_\delta(y)$. Now, if $x \in \mu_\delta(y)$, then $\mu y \leq \mu x < \delta$, and since $x \sqsubseteq y$, we have

$$d(x, y) \leq \mu x - \mu y < \varepsilon$$

which shows that $y \in \mu_\delta(y) \subseteq U$. Then we have proven that $\mu \to \sigma_D$. Consequently, $\ker \mu \subseteq \max D$. Now for the characterization of the kernel.

If the Scott and Lawson topologies agree at the top, then $\downarrow x \in \max F(D)$ whenever $x \in \max D$. Then it is clear that $\mu x = 0$ if $x \in \max D$. In the other direction, suppose that all maximal elements have measure zero and let $x \in \max D$. If $\downarrow x \subseteq C \in F(D)$, then $\bar{d}(\downarrow x, C) = 0$ since $\mu x = 0$. Consequently, $\downarrow x \in \max F(D)$. This is equivalent to saying that the Scott and Lawson topologies agree at the top [16]. □

Observe that, for arbitrary ω-continuous domains, the result makes no claims about the kernel. However, it does tell us that we can usually assume information content as expressible by a nonnegative real number.

Corollary 2.5.1 *If $\mu : P \to E$ is a Scott continuous mapping which induces the Scott topology on X, and E is an ω-continuous dcpo, then*

$$\exists \textit{ Scott continuous map } \mu : P \to [0, \infty)^*$$

which also induces the Scott topology on X.

proof Given $\mu : P \to E$ with $\mu \to \sigma_X$, the last result gives a Scott continuous map $\lambda : E \to [0, \infty)^*$ with $\lambda \to \sigma_E$. Since $\mu(X) \subseteq E$, the composition $\lambda\mu : P \to [0, \infty)^*$ is a Scott continuous map with $\lambda\mu \to \sigma_X$. □

Because $\mathcal{P}\omega$ is an ω-continuous Scott domain, Theorem 2.5.1 guarantees the existence of a measurement $\mu : \mathcal{P}\omega \to [0, \infty)^*$ with $\ker \mu = \{\mathbb{N}\}$.

Example 2.5.1 Cardinality is not precise enough to distinguish comparable infinite sets from one another, so we consider

$$|\cdot| : \mathcal{P}\omega \to [0,\infty)^*$$

given by

$$|x| = 1 - \sum_{n\in x} \frac{1}{2^{n+1}}.$$

This mapping is a measurement which induces the Scott topology everywhere whose kernel is exactly $\{\mathbb{N}\}$. First, the map is monotone

$$x \sqsubseteq y \Rightarrow 1 - \sum_{n\in x} \frac{1}{2^{n+1}} \geq 1 - \sum_{n\in y} \frac{1}{2^{n+1}},$$

and for an increasing sequence (x_n) in $\mathcal{P}\omega$,

$$\begin{aligned} |\bigcup x_n| &= 1 - \sum_{i\in\bigcup x_n} \frac{1}{2^{i+1}} \\ &= 1 - \lim_{n\to\infty} \sum_{i\in x_n} \frac{1}{2^{i+1}} \\ &= \lim_{n\to\infty} (1 - \sum_{i\in x_n} \frac{1}{2^{i+1}}) \\ &= \lim_{n\to\infty} |x_n|, \end{aligned}$$

so $|\cdot|$ is Scott continuous. The kernel is easy to compute since

$$|\mathbb{N}| = 1 - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = 1 - 1 = 0$$

and

$$|x| = 0 \Rightarrow \sum_{n\in x} \frac{1}{2^{n+1}} = 1 \Rightarrow x = \mathbb{N}.$$

To show that $|\cdot| \to \sigma_{\mathcal{P}\omega}$, suppose $k \ll x$ in $\mathcal{P}\omega$. Then k is a finite set we can assume nonempty, so it has a largest element $m \in \mathbb{N}$. If

$$s \sqsubseteq x \;\&\; |x| \leq |s| < |x| + \frac{1}{2^{m+1}},$$

then

$$\sum_{n\in s}\frac{1}{2^{n+1}} > \sum_{n\in x}\left(\frac{1}{2^{n+1}}\right) - \frac{1}{2^{m+1}}.$$

But this implies $k \sqsubseteq s$. For if $i \in k \setminus s$, then $s \sqsubseteq x \setminus \{i\}$, which means

$$\begin{aligned}\sum_{n\in s}\frac{1}{2^{n+1}} &\leq \sum_{n\in x\setminus\{i\}}\frac{1}{2^{n+1}}\\ &= \sum_{n\in x}\left(\frac{1}{2^{n+1}}\right) - \frac{1}{2^{i+1}}\\ &\leq \sum_{n\in x}\left(\frac{1}{2^{n+1}}\right) - \frac{1}{2^{m+1}}.\end{aligned}$$

Thus,

$$s \sqsubseteq x \ \& \ |s| < |x| + \frac{1}{2^{m+1}} \Rightarrow k \ll s,$$

which proves that $|\cdot|$ induces the Scott topology everywhere on $\mathcal{P}\omega$.

Example 2.5.2 By Lemma 2.2.3, the composition

$$[\mathbb{N} \to \mathbb{N}] \xrightarrow{\text{dom}} \mathcal{P}\omega \xrightarrow{|\cdot|} [0,\infty)^*$$

induces the Scott topology everywhere. This gives a measurement on the partial mappings $[\mathbb{N} \to \mathbb{N}]$ with $\ker|\text{dom}| = \max[\mathbb{N} \to \mathbb{N}]$.

The first part of Theorem 2.5.1 actually holds for any poset.

Example 2.5.3 Let P be a partially ordered set with a countable base $\{U_n : n \in \mathbb{N}\}$ for its Scott topology. The mapping

$$e : P \to \mathcal{P}\omega$$

$$e(x) = \{n \in \mathbb{N} : x \in U_n\}$$

is easily seen to be monotone and Scott continuous.

In fact, it induces the Scott topology everywhere. For if $U \subseteq P$ is a Scott open set around $x \in P$, there must be an n with $x \in U_n \subseteq U$. Now suppose that $a \sqsubseteq x$ and $\{n\} \ll e(a)$. Then $a \in U_n \subseteq U$. Hence, $x \in e_{\{n\}}(x) \subseteq U$.

Finally, by Lemma 2.2.3, the composition

$$P \xrightarrow{e} \mathcal{P}\omega \xrightarrow{|\cdot|} [0,\infty)^*$$

is a measurement which induces the Scott topology everywhere.

The last observation leads to a generalization of a well-known result [1].

Proposition 2.5.1 *For a partially ordered set whose Scott topology is second countable, the following are equivalent:*

(i) *Every increasing sequence has a supremum.*

(ii) *Every directed set has a supremum.*

In either case, every directed set contains an increasing sequence with the same supremum.

proof (i) $\Rightarrow$ (ii) We saw in Example 2.5.3 that the mapping $|e| : P \to [0, \infty)^*$ induces the Scott topology everywhere. By Lemma 2.2.5, it is strictly monotone. In addition, it is Scott continuous and every increasing sequence has a supremum. Now Theorem 2.2.1 applies. $\Box$

It is interesting perhaps that the last result holds even when the poset is *not* continuous.

2.6 The Contraction Principle

We now consider a contraction principle which illustrates some of the basic aspects of measurement studied earlier in this chapter.

Proposition 2.6.1 *Let $f : D \to D$ be a monotone map on a domain D with a measurement μ and suppose that $(\,\exists\, c < 1\,)(\,\forall\, x \in D\,)\, \mu f(x) \leq c \cdot \mu x$. If there is a point $x \sqsubseteq f(x)$, then*

$$x^\star = \bigsqcup_{n \geq 0} f^n(x) \in \max D$$

is a fixed point of f such that

$$(\,\forall\, a \ll x^\star\,)\ \bigsqcup_{n \geq 0} f^n(a) = x^\star.$$

Furthermore, the following are equivalent:

(i) *$x^\star$ is the unique fixed point of f on D.*

(ii) $(\,\forall\, x, y \in \mathrm{fix}(f)\,)(\,\exists\, z \in D\,)\ z \sqsubseteq x, y$.

proof First, for any $x \in D$ and any $n \geq 0$, $\mu f^n(x) \leq c^n \mu x$, as is easy to prove by induction. Given a point $x \sqsubseteq f(x)$, the monotonicity of f implies the sequence $(f^n(x))$ is increasing, while the continuity of μ allows us to compute

$$\mu(\bigsqcup f^n(x)) = \lim_{n\to\infty} \mu f^n(x) \leq \lim_{n\to\infty} c^n \mu x = 0.$$

Hence, $x^\star = \bigsqcup f^n(x) \in \ker \mu \subseteq \max D$. But the monotonicity of f also gives $x^\star \sqsubseteq f(x^\star)$. Hence, $x^\star = f(x^\star)$ is a fixed point of f. Next, let $a \sqsubseteq x^\star$. By the monotonicity of f,

$$(\forall n \geq 0)\ f^n(a) \sqsubseteq f(x^\star) = x^\star,$$

and since $\lim \mu f^n(a) = \mu x^\star = 0$, the fact that μ is a measurement yields

$$\bigsqcup f^n(a) = x^\star.$$

(i) $\Rightarrow$ (ii) is obvious. For (ii) $\Rightarrow$ (i), let $x_\star$ be any fixed point of f. By (ii), $(\,\exists\, z \in D\,)\ z \sqsubseteq x_\star, x^\star$. The same reasoning as above now shows that $\bigsqcup f^n(z) = x_\star = x^\star$. Thus, the fixed point $x^\star$ is unique. $\Box$

In general, such a mapping may have several fixed points, as can be seen by considering the identity map on an antichain. However, more often than not, each pair of elements in the kernel is bounded from below, in which case we obtain a unique fixed point.

Proposition 2.6.2 *Let D be a domain with a measurement μ such that*

$$(\,\forall\, x, y \in \ker \mu\,)(\,\exists\, z \in D\,)\ z \sqsubseteq x, y.$$

If $f : D \to D$ is a monotone map for which there is a constant $c < 1$ such that $\mu f(x) \leq c \cdot \mu x$, for all $x \in D$, and there is a point $x \in D$ with $x \sqsubseteq f(x)$, then

$$x^\star = \bigsqcup_{n\geq 0} f^n(x) \in \max D$$

is the unique fixed point of f on D.

Corollary 2.6.1 *Let D be a domain with a measurement μ and least element $\bot$. If $f : D \to D$ is monotone and $(\,\exists\, c < 1\,)(\,\forall\, x \in D\,)\, \mu f(x) \leq c \cdot \mu x$, then*

$$\bigsqcup f^n(\bot) \in \max D$$

is the unique fixed point of f on D.

One rarely encounters domains where the boundedness property of Proposition 2.6.2 fails to hold. For example, it is easy to see that $\mathbf{B}X$, $\mathbf{I}\mathbb{R}$, $\mathbf{U}X$ and $[S]$ all have this property even though none has a least element.

Example 2.6.1 Let $f : X \to X$ be a contraction on a complete metric space X with Lipschitz constant $c < 1$. The mapping $f : X \to X$ extends to a monotone map on the formal ball model $\bar{f} : \mathbf{B}X \to \mathbf{B}X$ given by

$$\bar{f}(x, r) = (fx, c \cdot r),$$

which satisfies

$$\pi \bar{f}(x, r) = c \cdot \pi(x, r),$$

where $\pi : \mathbf{B}X \to [0, \infty)^*$ is the standard measurement on $\mathbf{B}X$, $\pi(x, r) = r$. Now choose r so that $(x, r) \sqsubseteq \bar{f}(x, r)$. By Proposition 2.6.2, $\bar{f}$ has a unique fixed point which implies that f does also.

A nice feature of the contraction principle is that it applies to *any* domain with a measurement.

Example 2.6.2 A contraction $f : X \to X$ on a compact metric space X extends to a map on the upper space model $\bar{f} : \mathbf{U}X \to \mathbf{U}X$ given by

$$\bar{f}(K) = f(K).$$

This mapping is monotone by set theory, and if f has contraction constant $c < 1$, then

$$\text{diam}\, \bar{f}(K) \leq c \cdot \text{diam}\, K,$$

where $\text{diam} : \mathbf{U}X \to [0, \infty)^*$ is the standard measurement on $\mathbf{U}X$. The domain $\mathbf{U}X$ has a bottom element $\perp = X$, so the last corollary implies that $\bar{f} : \mathbf{U}X \to \mathbf{U}X$ and hence $f : X \to X$ has a unique fixed point.

However, the Banach contraction mapping theorem not only states that each contraction has a unique fixed point $x^\star$, it also tells us that $x^\star$ is an attractor: For any $x \in X$, the iterates $f^n(x)$ converge to $x^\star$. This is every bit as important as knowing $x^\star$ exists uniquely. We defer the proof of this fact until Chapter 5, where the topological structure of $\ker \mu$ is studied in detail.

Example 2.6.3 Consider the well-known functional

$$\phi : [\mathbb{N} \to \mathbb{N}] \to [\mathbb{N} \to \mathbb{N}]$$

defined by

$$\phi(f)(k) = \begin{cases} 1 & \text{if } k = 0, \\ kf(k-1) & \text{if } k \geq 1 \ \& \ k-1 \in \text{dom } f. \end{cases}$$

The natural measurement

$$\mu : [\mathbb{N} \to \mathbb{N}] \to [0, \infty)^*$$

$$\mu f = |\text{dom } f|$$

was studied in the last section. The mapping ϕ is easily seen to be monotone. Next we compute

$$\begin{aligned} \mu\phi(f) &= |\text{dom } \phi(f)| \\ &= 1 - \sum_{k \in \text{dom } \phi(f)} \frac{1}{2^{k+1}} \\ &= 1 - \left(\frac{1}{2^{0+1}} + \sum_{k-1 \in \text{dom } f} \frac{1}{2^{k+1}} \right) \\ &= 1 - \left(\frac{1}{2} + \sum_{k \in \text{dom } f} \frac{1}{2^{k+2}} \right) \\ &= \frac{1}{2} \left(1 - \sum_{k \in \text{dom } f} \frac{1}{2^{k+1}} \right) \\ &= \frac{\mu f}{2} \end{aligned}$$

which means ϕ is a contraction on the domain $[\mathbb{N} \to \mathbb{N}]$. By the contraction principle,

$$\bigsqcup_{n \in \mathbb{N}} \phi^n(\bot) = \text{fac}$$

is the unique fixed point of ϕ on $[\mathbb{N} \to \mathbb{N}]$, where $\bot$ is the function defined nowhere.

We will see other applications of the contraction principle later. However, there is one *potential* application which can be mentioned now. Advocates of the metric space approach to semantics site as one of its main advantages the existence of *unique* fixed points, as opposed to the *least* fixed points that domain theory provides. It would be nice to know if these two schools of thought can be unified by making use of a result like the one in this section.

2.7 Fixed Points of Nonmonotonic Maps

In the last section we used measurement to prove that certain monotone mappings have *unique* fixed points. Another advantage to measurement based reasoning is the ability to handle nonmonotonicity.

Proposition 2.7.1 *Let D be a domain with a measurement $\mu \to \sigma_D$. If $I \subseteq D$ is closed under directed suprema and $s : I \to I$ is a splitting whose measure*

$$\mu \circ s : I \to [0, \infty)^*$$

is Scott continuous between dcpo's, then

$$(\forall x \in I) \bigsqcup_{n \geq 0} s^n(x) \textit{ is a fixed point of } s.$$

Moreover, the set of fixed points $\mathrm{fix}(s) = \{x \in I : s(x) = x\}$ *is a dcpo.*

proof Let $x \in I$. By induction, $(s^n(x))$ is an increasing sequence in I. The set I is closed under directed suprema hence $\bigsqcup_{n \geq 0} s^n(x) \in I$. Because s is a splitting, $\bigsqcup_{n \geq 0} s^n(x) \sqsubseteq s(\bigsqcup_{n \geq 0} s^n(x))$, while the fact that $\mu \circ s$ and μ are both Scott continuous allows us to compute

$$\mu s(\bigsqcup_{n \geq 0} s^n(x)) = \lim_{n \to \infty} \mu s^{n+1}(x) = \mu(\bigsqcup_{n \geq 0} s^n(x)).$$

By Lemma 2.2.5, however, two comparable elements whose measures agree must in fact be equal. Hence,

$$s(\bigsqcup_{n \geq 0} s^n(x)) = \bigsqcup_{n \geq 0} s^n(x).$$

To show that fix(s) is a dcpo one need only prove closure under suprema of sequences in view of Theorem 2.2.1. The proof for sequences, however, uses the very same methods employed above and is entirely trivial. $\square$

Example 2.7.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map on the real line. Denote by $C(f)$ the subset of $\mathbf{I}\mathbb{R}$ where f changes sign, that is,

$$C(f) = \{[a, b] : f(a) \cdot f(b) \leq 0\}.$$

The continuity of f ensures that this set is closed under directed suprema, and the mapping

$$\mathrm{split}_f : C(f) \to C(f)$$

given by

$$\mathrm{split}_f[a, b] = \begin{cases} \mathrm{left}[a, b] & \text{if } \mathrm{left}[a, b] \in C(f); \\ \mathrm{right}[a, b] & \text{otherwise,} \end{cases}$$

is a splitting where $\mathrm{left}[a, b] = [a, (a + b)/2]$ and $\mathrm{right}[a, b] = [(a + b)/2, b]$. The measure of this mapping

$$\mu\, \mathrm{split}_f[a, b] = \frac{\mu[a, b]}{2}$$

is Scott continuous, so Proposition 2.7.1 implies that

$$\bigsqcup_{n \geq 0} \mathrm{split}_f^n[a, b] \in \mathrm{fix}(\mathrm{split}_f).$$

However, $\mathrm{fix}(\mathrm{split}_f) = \{[r] : f(r) = 0\}$, which means that iterating split_f is a scheme for calculating a solution of the equation $f(x) = 0$. This numerical technique is called *the bisection method.*

The major fixed point technique in classical domain theory, the Scott fixed point theorem, cannot be used to establish the correctness of the bisection method: split_f is only monotone in computationally irrelevant cases.

Proposition 2.7.2 *For a continuous selfmap $f : \mathbb{R} \to \mathbb{R}$ which has at least one zero, the following are equivalent:*

(i) *The splitting* split_f *is monotone.*

(ii) *The map f has a unique zero r and*

$$C(f) = \{[a, r] : a \leq r\} \cup \{[r, b] : r \leq b\}.$$

proof We prove (i) $\Rightarrow$ (ii). Let $\alpha < \beta$ be two distinct roots of f. Then by monotonicity of split_f,

$$\text{split}_f^n[\alpha, \beta] \sqsubseteq \text{split}_f[\beta] = [\beta],$$

for all $n \geq 0$. Then $[\alpha] = \bigsqcup \text{split}_f^n[\alpha, \beta] \sqsubseteq [\beta]$, which proves $\alpha = \beta$. Thus, f has a unique zero r.

Now let $[a, b] \in C(f)$ with $a < r < b$ and set $\delta = \max\{r - a, b - r\} > 0$. Then $r - \delta \leq a < b \leq r + \delta$. By the uniqueness of r,

$$f(r - \delta) \cdot f(a) > 0 \text{ and } f(b) \cdot f(r + \delta) > 0,$$

and since $[a, b] \in C(f)$, we have $\bar{y} := [r - \delta, r + \delta] \in C(f)$. For the very same reason, $\bar{x} := [r - \delta - \delta/2, r + \delta + \delta/4] \in C(f)$. But then we have $\bar{x} \sqsubseteq \bar{y}$ and

$$\text{split}_f \bar{x} = [r - \delta/8, r + \delta + \delta/4] \not\sqsubseteq [r - \delta, r] = \text{split}_f \bar{y},$$

which means split_f is not monotone if f changes sign on an interval which contains r in its interior. $\Box$

That is, if split_f is monotone, then in order to calculate the solution r of $f(x) = 0$ using the bisection method, we must first *know* the solution r. As further evidence of its applicability, notice that Proposition 2.7.1 also implies the Scott fixed point theorem for domains with measurements $\mu \to \sigma_D$.

Example 2.7.2 If $f : D \to D$ is a Scott continuous map on a domain D with a measurement $\mu \to \sigma_D$, then we consider its restriction to the set of points where it improves

$$I(f) = \{x \in D : x \sqsubseteq f(x)\}.$$

This evidently yields a splitting $f : I(f) \to I(f)$ on a dcpo with continuous measure. By Proposition 2.7.1,

$$(\forall x \in I(f)) \bigsqcup_{n \geq 0} f^n(x) \text{ is a fixed point of } f.$$

There are many ideas underlying the examples of this section that we will explore in the next chapter. For example, the attentive reader will have noticed that we did not *prove* that $C(f)$ is closed under directed suprema in Example 2.7.1. The reason this is true is that $C(f)$ is a closed set with respect to a topology intimately connected to measurement. The same holds for the set of improvements $I(f)$ of a Scott continuous map f.

2.8 Questions

(i) The natural measurements on $[S]$, Σ^∞, $\mathbf{I}\mathbb{R}$, $\mathcal{P}\omega$ and $[\mathbb{N} \to \mathbb{N}]$ can be thought of as "computable" since μx can be computed exactly in finite time provided that x is a basis element. Now imagine that we have a program which seeks to approximate some idealized object x whose measure μx is known. At iteration n, we are told the program will output a basis element $b_n \ll x$. What we need is some way to determine when b_n is close enough to x that we can stop the program. Before running the program, we choose a basis element $b \ll x$, and decide a good test for termination is that $b \ll b_n \ll x$. Because $b \ll x$, we know that

$$(\exists\, \varepsilon > 0)\, a \sqsubseteq x \;\&\; |\mu x - \mu a| < \varepsilon \Rightarrow b \ll a.$$

This $\varepsilon > 0$ is easily determined in the examples above given μx and b. Now we begin running the program. At iteration n, it outputs basis element $b_n \ll x$, and *because μb_n is computable exactly in finite time*, we can determine during execution whether or not $b \ll b_n$, by simply calculating μb_n and then measuring $|\mu x - \mu b_n|$. What is a "computable" measurement and why do we care?

(ii) When does a strictly monotone map $\mu : P \to [0, \infty)^*$ induce the Scott topology everywhere?

(iii) Is there a monotone map $\mu : D \to [0, \infty)^*$ on a domain, which induces the Scott topology everywhere, and is not Scott continuous?

(iv) Let (X, d) be a metric space. The set X^2 when ordered via

$$(a, b) \sqsubseteq (x, y) \Leftrightarrow d(a, x) + d(b, y) \leq d(a, b) - d(x, y)$$

is a poset and the metric $d : X^2 \to [0, \infty)^*$ is a monotone map between posets. In fact,

$$\ker d = \{(x, y) \in X^2 : d(x, y) = 0\} = \{(x, x) : x \in X\} = \max X^2.$$

When is X^2 a continuous poset? a domain? What kind of measurement is d?

Chapter 3

The μ Topology on a Domain

The μ topology arises naturally in the consideration of mappings which induce the Scott topology. It is a Hausdorff topology, larger than both the Scott and Lawson topologies, whose notion of limit seems ideally suited for computation. For example, every sequence with a μ limit has a supremum, even though such a sequence need not be directed; μ closed sets are closed under directed suprema, though they need not be lower sets. Properties like these often make the μ topology the natural choice when trying to formulate key computational ideas.

3.1 The μ Topology

In Chapter 2 it was observed that, for any monotone map $\mu : D \to E$, the collection

$$\{ \mu_\varepsilon(x) : x \in D, \varepsilon \in E \},$$

where $\mu_\varepsilon(x) = \{y \in D : y \sqsubseteq x \ \& \ \varepsilon \ll \mu y\}$, forms a basis for a topology on D. In particular, if we take μ to be the identity map on D, we obtain a topology with basis $\{ 1_\varepsilon(x) : x, \varepsilon \in D \} = \{ \twoheaduparrow x \cap \downarrow y : x, y \in D \}$.

Definition 3.1.1 On a continuous dcpo D, the μ *topology* has as a basis $\{ \twoheaduparrow x \cap \downarrow y : x, y \in D \}$. We denote this topology by μ_D in the same way that the Scott topology is written σ_D.

The identity map on D is easily seen to induce the Scott topology everywhere. It turns out that this is what determines the μ topology.

Theorem 3.1.1 (Invariance) *For a Scott continuous mapping $\mu : D \to E$ between domains, the following are equivalent:*

(i) *The mapping μ induces the Scott topology everywhere on D.*

(ii) $\{ \mu_\varepsilon(x) : x \in D, \varepsilon \in E \}$ *is a basis for the μ topology on D.*

proof First recall that $1_a(x) := \twoheaduparrow a \cap \downarrow x$ for all $a, x \in D$. (i) $\Rightarrow$ (ii): Let $a \in \mu_\varepsilon(x)$. By the continuity of μ, $(\exists\, z \ll a)\ a \in \twoheaduparrow z \subseteq \mu^{-1}(\twoheaduparrow \varepsilon)$. Then $a \in 1_z(a) \subseteq \mu_\varepsilon(x)$. Thus, each set $\mu_\varepsilon(x)$ is μ open. Now let U be a μ open set with $x \in U$. Then $(\exists\, a \ll x)\ 1_a(x) \subseteq U$. By (i), $\mu \to \sigma_D$, and since $x \in \twoheaduparrow a$, we know that $(\ \exists\, \varepsilon \in E\ )\ x \in \mu_\varepsilon(x) \subseteq \twoheaduparrow a$. But $\mu_\varepsilon(x) \subseteq \downarrow x$ so $x \in \mu_\varepsilon(x) \subseteq 1_a(x) \subseteq U$. Then $\{ \mu_\varepsilon(x) : x \in D, \varepsilon \in E \}$ is a basis for the μ topology on D. (ii) $\Rightarrow$ (i): Let U be a Scott open set around $x \in D$. Then $\exists\, a \ll x$ with $a \in U$. This gives $x \in 1_a(x) \subseteq U$. By (ii), $(\ \exists\, \varepsilon \in E\ )\ x \in \mu_\varepsilon(x) \subseteq 1_a(x) \subseteq U$. This proves $\mu \to \sigma_D$. $\square$

That is, no matter how we measure a domain, all measurements which induce the Scott topology everywhere place the μ topology on D.

Lemma 3.1.1 *On a continuous dcpo D,*

(i) *Every Lawson open set is μ open.*

(ii) *An upper set is μ open iff it is Scott open.*

(iii) *The upper set of a μ open set is Scott open.*

(iv) *Every μ closed set is closed under directed suprema.*

(v) $(\forall x \in D)\ x \in K(D)$ *iff $\{x\}$ is μ open.*

proof (i) Suppose $y \in \twoheaduparrow x \backslash \uparrow F$ where F is finite. Then no element below y is in $\uparrow F$. Since $\twoheaduparrow x$ is Scott open, there is an $a \ll y$ with $y \in \twoheaduparrow a \cap \downarrow y \subseteq \twoheaduparrow x \backslash \uparrow F$. Then basic Lawson open sets are μ open. (ii) Proposition 2.3.1 (iii) Apply Corollary 2.3.2 with $\mu = 1_D$. (iv) Let S be a directed subset of a μ closed set V and write $x = \bigsqcup_D S$. If $x \in D \backslash V$, then there is an $(a \ll x)\twoheaduparrow a \cap \downarrow x \subseteq D \backslash V$. By interpolation we must have $S \cap (D \setminus V) \neq \emptyset$. (v) Corollary 2.3.3 $\square$

A μ open set may be thought of intuitively as a Scott open set which is not necessarily an upper set.

Example 3.1.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map on the real line and consider

$$C(f) = \{ [a,b] \in \mathbf{I}\mathbb{R} : f(a) \cdot f(b) \leq 0 \}.$$

$C(f)$ is a μ closed subset of $\mathbf{I}\mathbb{R}$: Let $[a,b] \in \mathbf{I}\mathbb{R} \setminus C(f)$. Then $f(a) \cdot f(b) > 0$. By the continuity of f,

$$(\exists \varepsilon > 0)(s \in [a-\varepsilon, a] \ \&\ t \in [b, b+\varepsilon] \Rightarrow f(s) \cdot f(t) > 0),$$

which means $[a,b] \in \Uparrow[a-\varepsilon, b+\varepsilon] \cap \downarrow[a,b] \subseteq \mathbf{I}\mathbb{R} \setminus C(f)$. This proves that $\mathbf{I}\mathbb{R} \setminus C(f)$ is μ open and hence that $C(f)$ is μ closed. Finally, note that $C(f)$ is hardly ever Scott closed because it is rarely a lower set.

Proposition 3.1.1 *The μ topology on a continuous dcpo is zero-dimensional Tychonoff.*

proof For $p \in D$, let $x \in D \setminus \{p\}$. Since $x \neq p$, $(\exists a \ll x)\ p \notin \Uparrow a \cap \downarrow x$. Then $x \in \Uparrow a \cap \downarrow x \subseteq D \setminus \{p\}$. This proves that $\{p\}$ is μ closed which means that (D, μ_D) is a T_1 space. Now we claim that any set of the form $\Uparrow x \cap \downarrow y$ is also μ closed. Let $z \in D \setminus \Uparrow x \cap \downarrow y$. If $(\forall a \ll z)\ (\Uparrow a \cap \downarrow z) \cap (\Uparrow x \cap \downarrow y) \neq \emptyset$, then

$$(\forall a \ll z)\ a \ll y \ \&\ x \ll z$$

which means $x \ll z \sqsubseteq y$. This proves that $\Uparrow x \cap \downarrow y$ is μ closed. Then (D, μ_D) also has a basis of closed and open sets. Any zero-dimensional T_1 space is Hausdorff, regular and in fact Tychonoff (see 6.2 of [8]). □

Proposition 3.1.2 *For a continuous dcpo D, the following are equivalent:*

(i) *The Scott topology on D is first countable.*

(ii) *For all $x \in D$, there is an increasing sequence (x_n) in D, such that*

$$(\forall n)(x_n \ll x) \ \&\ \bigsqcup x_n = x.$$

(iii) *The μ topology on D is first countable.*

proof (i) $\Rightarrow$ (ii): Let $\{U_n\}$ be a countable basis of Scott open sets at $x \in D$. First, choose $a_n \in U_n$ with $a_n \ll x$ for each n. Next, set $x_1 = a_1$, and given any $a_n \sqsubseteq x_n \ll x$, choose $x_{n+1} \ll x$ with $x_n, a_{n+1} \sqsubseteq x_{n+1} \ll x$, using the directedness of $\Downarrow x$. The sequence (x_n) is increasing, so it has a supremum $\bigsqcup x_n$. Furthermore, this supremum belongs to each U_n since $a_n \sqsubseteq \bigsqcup x_n$.

Now suppose u is any upperbound of the sequence (x_n). Then $\bigsqcup x_n \sqsubseteq u$ so $u \in U_n$ for all n. But $\{U_n\}$ is a countable basis at x, so every open set around x contains u. Hence $x \sqsubseteq u$. Since x is above each x_n, we have $\bigsqcup x_n = x$. (ii) $\Rightarrow$ (iii): Using the sequence (x_n) from (ii), $\{\Uparrow x_n \cap \downarrow x : n \in \mathbb{N}\}$ is a countable basis at x w.r.t. the μ topology. (iii) $\Rightarrow$ (i): Let $\{U_n\}$ be a countable basis of μ open sets at x. Then $\{\uparrow U_n\}$ is a countable basis of Scott open sets around x since every Scott open set is μ open and the upper set of a μ open set is Scott open. Thus, (D, σ_D) is first countable at x. $\Box$

Recall that a subset B of a domain D is a *basis* for D iff $B \cap \Downarrow x$ contains a directed set with supremum x for each $x \in D$.

Proposition 3.1.3 *For a continuous dcpo D and any subset $B \subseteq D$, the following are equivalent:*

(i) *B is a basis for D.*

(ii) *B is a μ dense subset of D.*

proof (i) $\Rightarrow$ (ii): Let B be a basis for D and consider a basic μ open set $\Uparrow x \cap \downarrow y$. By interpolation, choose a with $x \ll a \ll y$. Since y is the supremum of a directed set contained in $\Downarrow y \cap B$, there is a $b \in B$ with $x \ll a \sqsubseteq b \sqsubseteq y$. Thus $(\Uparrow x \cap \downarrow y) \cap B \neq \emptyset$. This proves that B is a μ dense subset of D. (ii) $\Rightarrow$ (i): We will show that $B \cap \Downarrow x$ is directed with supremum x for each $x \in D$. Let $a \ll x$. By interpolation choose c with $a \ll c \ll x$. The set $\Uparrow c \cap \downarrow x$ is μ open so $(\exists b \in B)\ c \ll b \sqsubseteq x$ by the density of B. Hence,

$$(\ \forall a \ll x\ )(\ \exists b \in B\ )\ a \ll b \ll x.$$

Then the directedness of $\Downarrow x$ implies that $B \cap \Downarrow x$ is directed. Clearly, $\bigsqcup(B \cap \Downarrow x) \sqsubseteq x$. For the other inequality, if $a \ll x$, then $(\exists b \in B)\ a \ll b \ll x$. Thus $a \sqsubseteq b \sqsubseteq \bigsqcup(B \cap \Downarrow x)$. But a was arbitrary so $x \sqsubseteq \bigsqcup(B \cap \Downarrow x)$. $\Box$

Example 3.1.2 While every basis of a domain is Scott dense, a Scott dense set is not necessarily a basis: The collection of maximal elements $\max D$ is Scott dense, but

$$\begin{aligned} \max D \text{ is a basis for } D \ &\Leftrightarrow\ D = \max D \\ &\Leftrightarrow\ D \text{ has the discrete order.} \end{aligned}$$

Hence, the flat naturals $\mathbb{N}_\perp$ ($\mathbb{N}$ ordered discretely with a bottom attached) provide a counterexample.

Corollary 3.1.1 *For a continuous dcpo D, the following are equivalent:*

(i) *The Scott topology on D is second countable.*

(ii) *The continuous dcpo D is ω-continuous.*

(iii) *The μ topology on D is separable.*

proof For (i) = (ii) see Chapter III.4 of [10]. $\Box$

With respect to the μ topology, algebraicity of a domain is an extreme form of the Baire property.

Corollary 3.1.2 *A domain is algebraic iff the intersection of μ dense sets is μ dense.*

proof A continuous dcpo D is algebraic iff $K(D)$ is μ dense iff D has a smallest μ dense set [1]. $\Box$

Proposition 3.1.4 *If D is an ω-continuous dcpo and $|\max D| \geq |\mathbb{R}|$, then the μ topology on D is not normal.*

proof $\max D$ is a relatively discrete, closed subset of (D, μ_D). Since (D, μ_D) has a countable dense subset B and $|\max D| \geq |\mathbb{R}| \geq 2^{|B|}$, a lemma due to F.B. Jones shows that (D, μ_D) is not normal. (See for example Willard, p.100) $\Box$

Example 3.1.3 The μ topology on $\mathbf{I}\mathbb{R}$ is zero-dimensional, Tychonoff, separable and first countable. However, $|\max \mathbf{I}\mathbb{R}| = |\mathbb{R}|$, so it is not normal. Thus, it is also not compact, metrizable or second countable.

Proposition 3.1.5 *For a continuous dcpo D and any sequence (x_n), the following are equivalent:*

(i) *$x_n \rightarrow x$ in the μ topology.*

(ii) *$(\exists n)(\forall k \geq n)(x_k \sqsubseteq x)$ & $x_n \rightarrow x$ in the Scott topology.*

proof (i) $\Rightarrow$ (ii) Since all Scott open sets are μ open, it is clear that μ convergence implies convergence in the Scott topology. To see that x bounds most of the sequence, choose an approximation $a \ll x$. Since $\twoheaduparrow a \cap \downarrow x$ is μ open, it contains all but a finite number of the (x_n). (ii) $\Rightarrow$ (i) Let U be any μ open set around x. Then $\exists\, a \ll x$ with $\uparrow\!\!\!\uparrow a \cap \downarrow\! x \subseteq U$. Since $x_n \rightarrow x$ in the Scott topology, all but a finite number of the (x_n) are contained in $\uparrow\!\!\!\uparrow a$. The other assumption means that all but a finite number are bounded above by x. This proves μ convergence. $\Box$

In the presence of a measurement, μ limits are easy to compute.

Proposition 3.1.6 *If $\mu : D \rightarrow [0,\infty)^*$ is Scott continuous and $\mu \rightarrow \sigma_X$, then for any sequence (x_n) in D and any $x \in X$, the following are equivalent:*

(i) *$x_n \rightarrow x$ in the μ topology.*

(ii) *$(\exists n)(\forall k \geq n)(x_k \sqsubseteq x)$ & $\lim \mu x_n = \mu x$.*

proof (i) $\Rightarrow$ (ii): Let $\varepsilon > 0$ be arbitrary and set $\delta = \mu x + \varepsilon$. The map μ is Scott continuous, so $\mu_\delta(x)$ is a μ open set around x. Then by (i), $(\exists n)\; x_k \in \mu_\delta(x)$, for all $k \geq n$. Thus,

$$\begin{aligned} k \geq n &\Rightarrow\; x_k \sqsubseteq x \;\&\; \mu x_k < \delta \\ &\Rightarrow\; |\mu x_k - \mu x| = \mu x_k - \mu x < \varepsilon, \end{aligned}$$

which proves that $\lim \mu x_n = \mu x$. (ii) $\Rightarrow$ (i): We only need to prove that $x_n \rightarrow x$ in the Scott topology. Let U be a Scott open set around x. Since $\mu \rightarrow \sigma_X$, $(\exists\, \varepsilon > 0)\; x \in \mu_\varepsilon(x) \subseteq U$. Now by assumption $(\exists\, n_1)\; x_k \sqsubseteq x$ for $k \geq n_1$. Since $\lim \mu x_n = \mu x$,

$$(\exists n_2)(\forall k \geq n_2)\; |\mu x_k - \mu x| < \varepsilon - \mu x.$$

If $k \geq \max(n_1, n_2)$, then $\mu x_k < \varepsilon$, and so $x_k \in \mu_\varepsilon(x) \subseteq U$. $\Box$

Corollary 3.1.3 *If $x_n \rightarrow x$ in the μ topology, then*

$$(\exists n) \bigsqcup_{k \geq n} x_k = x$$

proof Let n be the least integer with $x_k \sqsubseteq x$ for $k \geq n$. Let u be an upper bound for $\{x_k : k \geq n\}$. Let U be any Scott open set containing x. Since $x_n \rightarrow x$ in the Scott topology, $(\exists\, m)\; x_k \in U$ for $k \geq m$. Then if $k \geq m, n$, $x_k \in U$, and so $u \in U$ since U is an upper set. Since every Scott open set containing x also contains u we must have $x \sqsubseteq u$. $\Box$

A sequence can converge in the μ topology without being directed.

Example 3.1.4 On the interval domain $\mathbf{I}\mathbb{R}$, set

$$x_n = \begin{cases} [\text{-}1/n, 0] & \text{if } n \text{ is odd} \\ [0, 1/n] & \text{if } n \text{ is even} \end{cases}$$

for $n \geq 1$. This sequence converges to $[0]$ in the μ topology but is *not* directed.

3.2 Mappings and Fixed Points

Definition 3.2.1 A function $f : D \to E$ is *μ-σ continuous* if it is topologically continuous as a mapping from (D, μ_D) to (E, σ_E). f is *μ continuous* if it is topologically continuous as a map from (D, μ_D) to (E, μ_E).

Proposition 3.2.1 *For a monotone map $f : D \to E$ between domains, the following are equivalent:*

(i) *f is μ-σ continuous.*

(ii) *f is μ continuous.*

(iii) *f is Scott continuous.*

proof (i) $\Rightarrow$ (iii) If f is a monotone μ-σ and $U \subseteq E$ is a Scott open set, $f^{-1}(U)$ is a μ open upper set, so $f^{-1}(U)$ is Scott open. (ii) $\Rightarrow$ (i) If f is μ continuous, then the inverse image of every Scott open set is μ open since Scott open sets are μ open. (iii) $\Rightarrow$ (ii) Let V be a μ open set and suppose that $x \in f^{-1}(V)$. Then since V is μ open, $(\exists\, y \ll f(x))\ \twoheaduparrow y \cap \downarrow f(x) \subseteq V$. By the Scott continuity of f,

$$y \ll f(x) = \bigsqcup_{a \ll x} f(a)$$

and so interpolation gives that $(\exists a \ll x) y \ll f(a)$. Then $\twoheaduparrow a \cap \downarrow x \subseteq f^{-1}(V)$, which proves that $f^{-1}(V)$ is μ open. $\square$

Corollary 3.2.1 *The class of μ-σ mappings includes the μ continuous mappings, Scott continuous mappings, and the Lawson continuous mappings.*

proof Every Lawson open set is μ-open. $\square$

Now we examine these general forms of continuity in more detail.

Proposition 3.2.2 *For a function $f : D \to E$ between domains, the following are equivalent:*

(i) *f is μ continuous.*

(ii) *f is μ-σ continuous and $(\forall x \in D)(\exists a \ll x)\, a \ll t \sqsubseteq x \Rightarrow f(t) \sqsubseteq f(x)$.*

proof More generally, suppose that $\mu : D \to F$ is any Scott continuous mapping into a domain F which induces the Scott topology everywhere. Then $\{\mu_\varepsilon(x) : x \in D, \varepsilon \in F\}$ is basis for the μ topology by invariance.

(i) $\Rightarrow$ (ii) The μ-σ continuity of f is clear. Now choose $y \ll f(x)$. Since $x \in f^{-1}(\Uparrow y \cap \downarrow f(x))$, and this set is μ open,

$$(\exists\, \varepsilon \in F)\, x \in \mu_\varepsilon(x) \subseteq f^{-1}(\Uparrow y \cap \downarrow f(x)).$$

This proves that

$$t \in \mu_\varepsilon(x) \Rightarrow f(t) \sqsubseteq f(x).$$

(ii) $\Rightarrow$ (i) Let $\Uparrow z \cap \downarrow y$ be a basic μ open set and suppose that $x \in f^{-1}(\Uparrow z \cap \downarrow y)$. Since f is μ-σ continuous, there is a μ open set U with $x \in U \subseteq f^{-1}(\Uparrow z)$. But since $f(x) \sqsubseteq y$, our other assumption means that

$$(\exists\, \varepsilon \ll \mu x)\, t \in \mu_\varepsilon(x) \Rightarrow f(t) \sqsubseteq f(x) \in \downarrow y.$$

Then $x \in U \cap \mu_\varepsilon(x) \subseteq f^{-1}(\Uparrow z \cap \downarrow y)$.

The proof of the statement in this proposition follows when we replace μ with the identity mapping on D. $\Box$

Though μ continuous maps do not necessarily preserve directed sets, they still preserve suprema.

Lemma 3.2.1 *If $f : D \to E$ is μ continuous and (x_n) is an increasing sequence in D, then*

$$(\exists k)\, f(\bigsqcup_{n\geq 1} x_n) = \bigsqcup_{n \geq k} f(x_n)$$

proof Let $x = \bigsqcup x_n$. Since $x_n \to x$ in the μ topology and f is μ continuous, $f(x_n) \to f(x)$ in the μ topology. But then $(\exists k)\ \bigsqcup_{n\geq k} f(x_n) = f(x)$. $\Box$

Proposition 3.2.3 *For a function $f : D \to E$ between domains and a measurement $\mu \to \sigma_D$, the following are equivalent:*

(i) *f is μ-σ continuous.*

(ii) $y \ll f(x) \Rightarrow (\exists\, \varepsilon > 0)(\ a \sqsubseteq x\ \&\ |\mu x - \mu a| < \varepsilon \Rightarrow y \ll f(a)\ ).$

proof (ii) $\Rightarrow$ (i) The set $\{a \in D : a \sqsubseteq x\ \&\ |\mu x - \mu a| < \varepsilon\}$ is μ open, for any choice of $x \in D$ and $\varepsilon > 0$, since it may be written as $\mu_\delta(x)$, where $\delta = \mu x + \varepsilon$. (i) $\Rightarrow$ (ii) If $y \ll f(x)$, then by μ-σ continuity, $(\exists \delta > 0) x \in \mu_\delta(x) \subseteq f^{-1}(\mathord{\uparrow\!\!\uparrow} y)$. Choosing $\varepsilon = \delta - \mu x > 0$ finishes the proof. □

Lemma 3.2.2 *If $f : D \to E$ is a function between domains, then*

(i) *f is μ-σ continuous at any x where $f(x) = \bot$, and*

(ii) *f is μ continuous at any $x \in K(D)$.*

proof For (i), $y \ll f(x) \Rightarrow y = \bot \Rightarrow (\ \forall\, a \ll x\ )(\ a \ll t \sqsubseteq x \Rightarrow y \ll f(t)\ )$. To see (ii), the set $\{x\}$ is μ open since $x \in K(D)$. □

Example 3.2.1 Let S be a finite linear order and consider the domain $[S]$ of lists over S. The μ topology on $[S]$ is discrete, so every function defined on $[S]$ is μ continuous. For example,

$$\text{sort} : [S] \to [S],$$

which takes a list and sorts it, is a μ continuous map which is not monotone.

Example 3.2.2 The mapping

$$\text{left} : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$$

$$\text{left}[a, b] = [a, \frac{a+b}{2}]$$

is μ-σ continuous: Let $x_n \to x$ in the μ topology on $\mathbf{I}\mathbb{R}$. We need to show that $\text{left}(x_n) \to \text{left}(x)$ in the Scott topology. We may assume that $\bigsqcup x_n = x$. Writing $x_n = [a_n, b_n]$ and $x = [a, b]$, we see that μ convergence yields

$$a_n \le a \le b \le b_n\ \&\ \lim_{n\to\infty} (b_n - a_n) = b - a.$$

Thus, $\lim a_n = a$ and $\lim b_n = b$. Now suppose that $I \ll \text{left}(x)$. Then $[a, (a+b)/2] = \text{left}(x) \subseteq \text{int}(I)$. Thus, for large enough n, we have

$$a_n \in \text{int}(I) \ \& \ \frac{a_n + b_n}{2} \in \text{int}(I),$$

which means $I \ll \text{left}(x_n)$. Consequently, $\text{left}(x_n) \to \text{left}(x)$ in the Scott topology. This proves μ-σ continuity since $\mathbf{I}\mathbb{R}$ is first countable. Suprisingly, this map is *not* μ continuous: If left were μ continuous at $[0] \in \mathbf{I}\mathbb{R}$, then

$$(\exists\, \varepsilon > 0)\ I \in \mu_\varepsilon([0]) \Rightarrow \text{left}(I) \sqsubseteq \text{left}[0] = [0].$$

But for the interval $I_\varepsilon = [\text{-}\varepsilon/3, \varepsilon/6]$, we have

$$\mu I_\varepsilon = \frac{\varepsilon}{2} < \varepsilon \ \& \ I_\varepsilon \sqsubseteq [0],$$

which means $I_\varepsilon \in \mu_\varepsilon([0])$. However,

$$0 \notin \text{left}(I_\varepsilon) = [\text{-}\frac{\varepsilon}{3}, \text{-}\frac{\varepsilon}{12}],$$

which gives the contradiction. This also shows that left : $\mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ is not monotone or Scott continuous.

Unlike a μ continuous mapping, a μ-σ may not preserve suprema.

Example 3.2.3 Consider the map left : $\mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ of the last example, along with the increasing sequence $x_n = [\text{-}\frac{1}{3n}, \frac{1}{6n}]$, for $n \geq 1$. Clearly, $\bigsqcup x_n = [0]$, but the sequence $\text{left}(x_n) = [\text{-}\frac{1}{3n}, \text{-}\frac{1}{12n}]$ is not even bounded from above!

Lemma 3.2.3 *If $f : D \to E$ is μ-σ continuous, (x_n) is an increasing sequence in D, and $(f(x_n))$ is an increasing sequence in E, then*

$$f(\bigsqcup x_n) \sqsubseteq \bigsqcup f(x_n)$$

proof First, let (x_n) be an increasing sequence in D for which $(f(x_n))$ is also increasing. Since $x_n \to \bigsqcup x_n$ in the μ topology, μ-σ continuity of f implies that $f(x_n) \to f(\bigsqcup x_n)$ in the Scott topology. Since $(f(x_n))$ is increasing, we have $f(\bigsqcup x_n) \sqsubseteq \bigsqcup f(x_n)$. $\square$

There are also fixed point theorems available for μ and μ-σ continuous maps.

Theorem 3.2.1 *If $f : D \to D$ is μ-σ continuous and $I(f) = \{x : x \sqsubseteq f(x)\}$ is a dcpo, then*

$$(f^n(x)) \textit{ increasing} \Rightarrow \bigsqcup f^n(x) \in \text{fix}(f)$$

for all $x \in D$. Furthermore, $\text{fix}(f)$ is a dcpo.

proof First, let (a_n) be an increasing sequence in D for which $(f(a_n))$ is also increasing. Then μ-σ continuity implies that $f(\bigsqcup a_n) \sqsubseteq \bigsqcup f(a_n)$. If in addition we know that $a_n \in I(f)$ for all n, then since $I(f)$ is a dcpo, $\bigsqcup a_n \in I(f)$, so $\bigsqcup a_n \sqsubseteq f(\bigsqcup a_n)$. Now setting $a_n = f^n(x)$ for $n \geq 0$ yields

$$\bigsqcup_{n\geq 0} f^n(x) \sqsubseteq f(\bigsqcup_{n\geq 0} f^n(x)) \sqsubseteq \bigsqcup_{n\geq 0} f(f^n(x)) = \bigsqcup_{n\geq 0} f^n(x)$$

and so $\bigsqcup f^n(x) \in \text{fix}(f)$. Now write $I^*(f) = \{x : f(x) \sqsubseteq x\}$. This set is always μ closed provided that f is μ-σ continuous: Let $x \in D \setminus I^*(f)$. Then $(\exists\, y \ll f(x))\, y \not\sqsubseteq x$. By μ-σ continuity,

$$(\exists\, a \ll x)\, a \ll t \sqsubseteq x \Rightarrow y \ll f(t).$$

Consequently, $x \in \twoheaduparrow a \cap \downarrow x \subseteq D \setminus I^*(f)$, which proves that $I^*(f)$ is μ closed. Thus, $\text{fix}(f) = I(f) \cap I^*(f)$ is a dcpo as the intersection of dcpo's. □

We normally prove $I(f)$ is a dcpo by showing that it is μ closed.

Corollary 3.2.2 *If $f : D \to D$ is μ continuous, then*

$$(f^n(x)) \textit{ increasing} \Rightarrow \bigsqcup f^n(x) \in \text{fix}(f)$$

for all $x \in D$ and $\text{fix}(f)$ is a dcpo.

proof The set of inputs that f improves $I(f) = \{x : x \sqsubseteq f(x)\}$ is μ closed: Let $x \in D \setminus I(f)$. Then $(\exists\, b \ll x)\, b \not\sqsubseteq f(x)$. By μ continuity,

$$(\exists\, a \ll x)\, a \ll t \sqsubseteq x \Rightarrow f(t) \sqsubseteq f(x).$$

By directedness of $\twoheaddownarrow x$, $(\exists\, c)\, a, b \sqsubseteq c \ll x$. Then $x \in \twoheaduparrow c \cap \downarrow x \subseteq D \setminus I(f)$. Finally, all μ continuous maps are μ-σ continuous. □

Corollary 3.2.3 *If $f : D \to D$ is Scott continuous, then*

$$(\forall x \in D)\, x \sqsubseteq f(x) \Rightarrow \bigsqcup f^n(x) \in \text{fix}(f)$$

and $\text{fix}(f)$ is a dcpo.

proof Every Scott continuous map is μ continuous and the sequence $(f^n(x))$ is increasing by monotonicity. $\square$

Corollary 3.2.4 *If $s : D \to D$ is a μ-σ continuous splitting, then*

$$(\forall x \in D) \; \bigsqcup s^n(x) \in \mathrm{fix}(s),$$

and fix(s) *is a dcpo.*

proof $I(s) = \{x : x \sqsubseteq s(x)\} = D$ is μ closed. $\square$

There is something of a converse for splittings.

Lemma 3.2.4 *A splitting is μ-σ continuous at a fixed point.*

proof Let $s : D \to D$ be a splitting with a fixed point $s(x) = x$. If $y \ll s(x)$, then since $\Uparrow\!y$ is μ open, $(\exists a \ll x)(\, a \ll t \sqsubseteq x \Rightarrow y \ll t\,)$. But s is a splitting so $y \ll t \sqsubseteq s(t)$ for $a \ll t \sqsubseteq x$. $\square$

Thus, in addition to compact elements, splittings are also μ-σ continuous at maximal elements.

Example 3.2.4 Every splitting $s : \Sigma^\infty \to \Sigma^\infty$ on the Cantor set model is μ-σ continuous since every element is either maximal or compact.

In all the fixed point theorems we have considered thus far, we have required the sequence $(f^n(x))$ increasing. The next example shows what can happen if we do not assume this.

Example 3.2.5 On the Cantor set model Σ^∞ with standard measurement

$$\mu s = \frac{1}{2^{|s|}}$$

consider $f : \Sigma^\infty \to \Sigma^\infty$ defined by

$$f(s) = \begin{cases} 0 & \text{if } s = \varepsilon \\ 0 \cdot s & \text{if first}(s) = 1 \\ 1 \cdot s & \text{if first}(s) = 0 \end{cases}$$

where first : $\Sigma^\infty \setminus \{\varepsilon\} \to \{0, 1\}$ is the first bit of a nonempty string in Σ^∞. The mapping f has each of the following properties:

(a) f is μ continuous but not monotone.

(b) $I(f) = \{\varepsilon\}$ is μ closed.

(c) f has no fixed points.

For the proof of (a), each nonempty string $s \in \Sigma^\infty$ may be written as $s = \text{first}(s) \cdot \text{rest}(s)$, where $\text{rest}(s)$ is the string s with its first bit removed. Let $y \ll f(x)$ with $x \in \max \Sigma^\infty$. Then y is a finite string (which may be assumed nonempty) and $\text{rest}(y) \ll x$. Thus, there is $\varepsilon > 0$ such that

$$\begin{aligned} s \in \mu_\varepsilon(x) &\Rightarrow \text{rest}(y) \ll s \\ &\Rightarrow y \ll \text{first}(y) \cdot s \\ &\Rightarrow y \ll \text{first}(f(x)) \cdot s \\ &\Rightarrow y \ll \text{first}(f(s)) \cdot s = f(s) \end{aligned}$$

where the last implication follows from $s \sqsubseteq x \Rightarrow \text{first}(f(s)) = \text{first}(f(x))$. This establishes the μ-σ continuity of f since all maps are μ-σ at compact elements. Finally,

$$x \neq \varepsilon \ \&\ x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y),$$

which finishes the proof that f is μ continuous. Removing ε from Σ^∞ yields a Scott continuous map on an algebraic domain with $I(f) = \emptyset$.

3.3 The Measure of a Mapping on a Domain

We have seen several times now that the existence of a measurement on a domain simplifies reasoning about the μ topology. For example, Proposition 3.1.6 rephrases μ convergence as being something quite intuitive: A sequence (x_n) converges to x iff most of the x_n are below x and the measures μx_n converge to μx. It is natural then to expect, for example, that continuity of functions may also be partially expressed in terms of measurement. What comes as a surprise, however, is that the use of measurement as a clarifying device for the μ topology leads to *generalizations* of previous results.

By the measure of a mapping $f : D \to E$, we mean the composition

$$D \xrightarrow{f} E \xrightarrow{\mu} [0, \infty)^*$$

where μ is a measurement on E. We say that f has *monotone measure* if μf is a monotone map between domains and that f has *Scott continuous measure* if μf is Scott continuous. Similar phrases are used for μ and μ-σ continuity.

3.3.1 Mappings

Proposition 3.3.1 *For a function $f : D \to E$ into a domain E with a measurement $\mu \to \sigma_E$, the following are equivalent:*

(i) *f is μ continuous.*

(ii) *μf is μ continuous and $(\forall x \in D)(\exists a \ll x)\, a \ll t \sqsubseteq x \Rightarrow f(t) \sqsubseteq f(x)$.*

proof (ii) $\Rightarrow$ (i): Let U be a μ open subset of E and suppose $x \in f^{-1}(U)$. By (ii), $(\exists\, a \ll x\,)\; a \ll t \sqsubseteq x \Rightarrow f(t) \sqsubseteq f(x)$. By the μ continuity of μf,

$$(\exists\, a_n \ll x\,)\, \uparrow\!\!\!\!\uparrow a_n \cap \downarrow\! x \subseteq (\mu f)^{-1}([\mu f(x), \mu f(x) + 1/n)) \;\cap\; \uparrow\!\!\!\!\uparrow a.$$

Now suppose, by way of contradiction, that for each $n \geq 1$, we can find x_n with $a_n \ll x_n \sqsubseteq x$ and $x_n \notin f^{-1}(U)$. First, since $a \ll x_n \sqsubseteq x$, $f(x_n) \sqsubseteq f(x)$, for all n. Next, we have the inequality $\mu f(x) \leq \mu f(x_n) < \mu f(x) + 1/n$, for all n, which means that

$$\lim_{n\to\infty} \mu f(x_n) = \mu f(x) \;\&\; (\forall n)\; f(x_n) \sqsubseteq f(x).$$

Hence $f(x_n) \to f(x)$ in the μ topology. Then for all but a finite number of the x_n, we must have $f(x_n) \in U$, which contradicts $x_n \notin f^{-1}(U)$. □

Corollary 3.3.1 *Let $f : D \to E$ be a function into a domain E with a measurement $\mu \to \sigma_E$ and suppose that μf is monotone. Then the following are equivalent:*

(i) *f is μ continuous.*

(ii) *μf is Scott continuous & $(\forall x \in D)(\exists a \ll x) a \ll t \sqsubseteq x \Rightarrow f(t) \sqsubseteq f(x)$.*

proof (i) $\Rightarrow$ (ii): The measure μf is a monotone map which is also the composition of μ continuous maps. Hence, it is a monotone μ continuous map which is to say that it is Scott continuous. □

Corollary 3.3.2 *If $f : D \to E$ is a function into a domain E with a measurement $\mu \to \sigma_E$, the following are equivalent:*

(i) *f is Scott continuous.*

(ii) *f is monotone and μf is Scott continuous.*

proof (ii) $\Rightarrow$ (i): By the last corollary, f is both monotone and μ continuous. But a monotone map is μ continuous iff it is Scott continuous. □

3.3.2 Fixed Points

We know that a monotone μ-σ continuous function has the fixed point property because such a mapping is Scott continuous. Unexpectedly, the same is true of a μ-σ continuous mapping whose *measure* is monotone.

Proposition 3.3.2 *Let D be a domain with a measurement $\mu \to \sigma_D$. If $f : D \to D$ is is μ-σ continuous and has monotone measure, then*

$$(f^n(x)) \textit{ increasing} \Rightarrow \bigsqcup f^n(x) \in \mathrm{fix}(f)$$

for all $x \in D$. Furthermore, $\mathrm{fix}(f)$ *is a dcpo.*

proof Since μ is Scott continuous and f is μ-σ continuous, μf is also μ-σ continuous. But this means μf is a monotone μ-σ, so it is Scott continuous. Now let (x_n) be an increasing sequence for which $(f(x_n))$ is also increasing. As pointed out earlier, $f(\bigsqcup x_n) \sqsubseteq \bigsqcup f(x_n)$, by μ-σ continuity. Since μf is Scott continuous,

$$\mu(f(\bigsqcup x_n)) = (\mu f)(\bigsqcup x_n) = \lim_{n\to\infty} \mu f(x_n) = \mu(\bigsqcup f(x_n)),$$

and because $\mu \to \sigma_D$, μ is strictly monotone, so $f(\bigsqcup x_n) = \bigsqcup f(x_n)$. Setting $x_n = f^n(x)$ now shows that $\bigsqcup f^n(x)$ is a fixed point of f. Now let $S \subseteq \mathrm{fix}(f)$ be directed. Since $\mu \to \sigma_D$, S contains an increasing sequence (x_n) whose supremum is $\bigsqcup S$. Since (x_n) and $(f(x_n))$ are both increasing,

$$f(\bigsqcup x_n) = \bigsqcup f(x_n) = \bigsqcup x_n = \bigsqcup S \in \mathrm{fix}(f),$$

which means that $\mathrm{fix}(f)$ is a dcpo. $\Box$

As another example, recall Corollary 3.2.4, which states that any μ-σ continuous splitting has the fixed point property. Suprisingly, the result still holds if we assume that only the *measure* of the splitting is μ-σ continuous.

Proposition 3.3.3 *Let D be a domain with a measurement $\mu \to \sigma_D$. If $s : D \to D$ is a splitting with μ-σ continuous measure, then*

$$(\forall x \in D) \bigsqcup_{n\geq 0} s^n(x) \in \mathrm{fix}(s).$$

In addition, the set $\mathrm{fix}(s)$ *is a dcpo.*

proof Let $a_n \sqsubseteq s(a_n) \sqsubseteq a_{n+1}$ be a sequence in D. Then $a_n \to \bigsqcup a_n$ in the μ topology so $\mu s(a_n) \to \mu s(\bigsqcup a_n)$ in the Scott topology, as s has μ-σ measure. But

$$\bigsqcup s(a_n) = \bigsqcup a_n \sqsubseteq s(\bigsqcup a_n),$$

so $\mu s(a_n) \geq \mu s(\bigsqcup a_n)$. Thus, $\mu s(a_n) \to \mu s(\bigsqcup a_n)$ in the μ topology on $[0,\infty)^*$. This in turn implies that

$$\mu(\bigsqcup a_n) = \mu(\bigsqcup s(a_n)) = \lim_{n\to\infty} \mu s(a_n) = \mu s(\bigsqcup a_n).$$

But by the strict monotonicity of μ we must have $\bigsqcup a_n = s(\bigsqcup a_n)$. The result is now a trivial consequence of what we have shown by setting $a_n = s^n(x)$. □

There is magic in the last result: Plenty of splittings have μ-σ measure but are not actually μ-σ continuous.

Example 3.3.1 Consider the continuous dcpo $D = \downarrow[0] \subseteq \mathbf{I}\mathbb{R}$ whose natural measurement is simply the restriction of the length map on $\mathbf{I}\mathbb{R}$. Now consider the splitting

$$s : D \to D$$

given by

$$s[a,b] = \begin{cases} \text{left}[a,b] & \text{if } 0 \in \text{left}[a,b]; \\ \text{right}[a,b] & \text{otherwise.} \end{cases}$$

This function arises when we use the bisection method to compute the unique root of $f(x) = x$ on $\mathbb{R}$. Suprisingly, it is not μ-σ continuous. Consider, for example, the interval $x = [-1,1]$. If s were μ-σ continuous at x, then since $[-2,1/2] \ll s(x)$,

$$(\exists\, \varepsilon > 0)\; a \sqsubseteq x \;\&\; |\mu x - \mu a| < \varepsilon \Rightarrow [-2,1/2] \ll s(a).$$

However, if $a = [-1-\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{4}]$, then $s(a) = [-\frac{\varepsilon}{8}, 1+\frac{\varepsilon}{4}]$, which gives the desired contradiction. Then s is a splitting with Scott continuous measure which is not μ-σ continuous. Observe that the Cantor set model may be used to construct μ-σ continuous splittings with nonmonotonic measure!

Thus, when the μ topology is expressible in terms of a measurement, we simply obtain stronger results.

3.3.3 Domain Theory and the Identity Map

We now spend a little time considering what took place in the fixed point theorems of the last subsection in relation to traditional domain theory. This section is merely designed to illustrate a particular philosophy: We are not so concerned with the value of its mathematical content.

To begin, lets think about what it means for a process $f : D \to D$ to be monotone. One possible view is that it means $f(y)$ is at least as *informative* as $f(x)$ whenever $x \sqsubseteq y$. The question is, what does *informative* mean? The extent to which $f(x)$ and $f(y)$ are informative is determined by their respective information contents. Intuitively, the "amount" of information each contains. With this realization, we see that a mechanism for determining the information content of an object in a domain is required.

To this end, consider an "abstract measurement" $\mu : D \to E$, which maps into some domain E, and induces the Scott topology everywhere. We can now express the information content of $f(x)$ as $\mu f(x)$. Our original view of the monotonicity of f now reads

$$x \sqsubseteq y \Rightarrow \mu f(x) \sqsubseteq \mu f(y),$$

that is, μf is monotone.

Since its inception, Domain theory has been based on the premise that the information content of $f(x)$ is $f(x)$ itself. In the present discussion, we can see this amounts to assuming that the appropriate notion of measurement on a domain is the identity mapping. Hence, the monotonicity of a process f can only be expressed by requiring that f itself is monotone. However, by allowing the choice of measurement to vary, that is, weakening our notion of information content, we are able to express the monotone nature of a process without actually requiring it monotone. This is what was done, for example, in Proposition 3.3.2.

Definition 3.3.1 For a continuous dcpo D and a Scott continuous mapping $\mu : D \to E$ with $\mu \to \sigma_D$,

$$[D \to D](\mu) := \{\ f \mid f : D \to D \text{ is } \mu\text{-}\sigma \text{ continuous } \&\ \mu f \text{ is monotone } \}.$$

We think of $[D \to D](\mu)$ as a theory about the selfmaps on D.

Proposition 3.3.4 *Any theory $[D \to D](\mu)$ contains all of the Scott continuous mappings. $[D \to D](1_D)$ contains only the Scott continuous mappings. Hence, $[D \to D](1_D)$ is the smallest theory about mappings on D.*

What is important is that changing measurements usually yields more mappings than does the standard theory $[D \to D](1_D) = [D \to D]$.

Example 3.3.2 Let (X, d) be a complete metric space and let $f : X \to X$ be a continuous map. For a constant $k > 0$, consider

$$\overline{f} : \mathbf{B}X \to \mathbf{B}X$$

$$\overline{f}(x, r) = (fx, k \cdot r)$$

where $\mathbf{B}X$ is the formal ball model [7] of X whose natural measurement is

$$\pi : \mathbf{B}X \to [0, \infty)^*$$

$$\pi(x, r) = r.$$

Then $\overline{f}$ is μ-σ continuous and has monotone measure: Let $(y, s) \ll \overline{f}(x, r)$. Choose w such that $kr < w < s - d(y, fx)$. Then $fx \in B_{s-w}(y)$, so by continuity of f,

$$(\exists\, \varepsilon > 0)\; B_\varepsilon(x) \subseteq f^{-1}(B_{s-w}(y)).$$

Now let $(z, t) \sqsubseteq (x, r)$ with $r \leq t < \min\{\frac{w}{k}, r + \varepsilon\}$. Then $d(x, z) \leq t - r < r + \varepsilon - r = \varepsilon$, which means that $d(y, fz) < s - w$, so $(y, s) \ll (fz, w)$. Since $t < \frac{w}{k}$,

$$(y, s) \ll (fz, k \cdot t) = \overline{f}(z, t).$$

Observe then that *all* continuous maps on X have μ-σ extensions to $\mathbf{B}X$: Only the Lipschitz mappings on X are known to have Scott continuous extensions (if f is a Lipschitz mapping, we can use the Lipschitz constant as our value for k, and $\overline{f}$ will be monotone). This shows that $[\mathbf{B}X \to \mathbf{B}X](\pi)$ and $[\mathbf{B}X \to \mathbf{B}X]$ are very different.

Example 3.3.3 Every splitting on the Cantor set model Σ^∞ is a μ-σ. For example, right multiplication by 0,

$$r_0 : \Sigma^\infty \to \Sigma^\infty$$

$$r_0(s) = \begin{cases} s \cdot 0 & \text{if } s \text{ finite} \\ s & \text{otherwise} \end{cases}$$

is μ-σ continuous with Scott continuous measure but is not monotone.

Example 3.3.4 The mapping

$$\text{right} : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$$

$$\text{right}[a, b] = [\frac{a+b}{2}, b]$$

is a μ-σ continuous splitting with Scott continuous measure which is not monotone.

We conclude the following: As we weaken our standard notion of information content from 1_D to μ, the Scott fixed point theorem generalizes to Proposition 3.3.2, while Corollary 3.2.4 turns in to Proposition 3.3.3.

3.4 Subdomains and Clopen Sets

In this section, we present a technique which enables one to construct μ continuous mappings on a domain with ease: We simply paste together μ continuous mappings defined on subdomains.

Definition 3.4.1 A subset V of a domain $(D, \sqsubseteq)$ is called a *subdomain* if $(V, \sqsubseteq)$ is a continuous dcpo such that

(i) For every directed $S \subseteq V$, $\bigsqcup_V S = \bigsqcup_D S$, and

(ii) For all $x, y \in V$, $x \ll_V y \Leftrightarrow x \ll_D y$.

For clarity, we begin with an example of a subset of a domain which is a continuous dcpo but *not* a subdomain.

Example 3.4.1 Let $\mathbb{A}$ be the subset of $\mathcal{P}\omega$ given by

$$\{A_n : n \geq 0\}$$

where $A_n = \mathbb{N} \setminus n$ for all $n \geq 0$. This set has many interesting properties:

(i) $(\mathbb{A}, \subseteq)$ is a continuous dcpo: It is an antichain, so the order is discrete.

(ii) $\mathbb{A}$ is not a subdomain of $\mathcal{P}\omega$: Every element of $(\mathbb{A}, \subseteq)$ is compact, but none is compact in $\mathcal{P}\omega$.

(iii) $\mathbb{A}$ is not μ open: It does not contain any approximations (finite sets) of any of its elements.

(iv) $\mathbb{A}$ is not μ closed: The sequence (A_n) converges to $\mathbb{N}$ in the μ topology on $\mathcal{P}\omega$, but $\mathbb{N} \notin \mathbb{A}$. To see this convergence, use the natural measurement $|\cdot|$ on $\mathcal{P}\omega$ from Example 2.5.1, and note that

$$|A_n| = 1 - \sum_{i \in A_n} \frac{1}{2^{i+1}} = \frac{1}{2^{n+1}} \to |\mathbb{N}| = 0.$$

Since $A_n \sqsubseteq \mathbb{N}$ for all n, Proposition 3.1.6 gives $A_n \to \mathbb{N}$.

(v) $\mathbb{A}$ is closed under directed suprema: The only directed subsets are one point sets.

Subdomains have more structure than random sets closed under directed suprema.

Proposition 3.4.1 *Every subdomain is μ closed.*

proof Let V be a subdomain of D. To show $D \setminus V$ μ open, let $x \in D \setminus V$. If $D \setminus V$ is not μ open, then for all $a \ll_D x$, there is $b \in V$ with $a \ll_D b \sqsubseteq x$. Because V is a subdomain, the set $\twoheaddownarrow_V b$ is a directed subset of $\twoheaddownarrow_D b$ whose supremum in D is b. Then by interpolation in D,

$$(\exists\, z \in \twoheaddownarrow_V b)\; a \ll_D z \ll_D b,$$

where $z \ll_D b$ holds because $\ll_V \subseteq \ll_D$ in a subdomain. The set of all such z forms a directed subset of V with supremum x. Since V is closed under directed suprema, this puts $x \in V$, which is a contradiction. $\Box$

At first glance, subdomains may also seem to be open subsets in the μ topology, if one considers the fact that they contain arbitrarily close approximations of each of their elements. The problem with this idea is that μ open sets are locally convex. The same cannot be said of a subdomain.

Example 3.4.2 Let V be the subdomain of $\mathbb{N}^\infty$ given by

$$V = \{2n : n \in \mathbb{N}\} \cup \{\infty\}.$$

The set V is μ closed but not μ open: It contains no μ open set around ∞.

Before continuing, we need to emphasize a simple but important result.

Lemma 3.4.1 *Let V be a μ open subset of a domain D. If $S \subseteq D$ is a directed subset with $\bigsqcup_D S \in V$, then V contains a directed subset of S with the same supremum in D.*

proof We write $x = \bigsqcup_D S \in V$. Because V is μ open, there is an element $a \ll_D x$ with $\Uparrow_D a \cap \downarrow_D x \subseteq V$. The set

$$S_x := S \cap \Uparrow_D a \cap \downarrow_D x \subseteq V$$

is nonempty using interpolation in D, and directed by the directedness of S. We show it has supremum x. Let u be any upperbound for S_x. If $s \in S$, choose an element $s_1 \in S_x$, and use the directedness of S to obtain $s_2 \in S$ with $s, s_1 \sqsubseteq s_2$. But this gives

$$a \ll_D s_1 \sqsubseteq s_2 \sqsubseteq \bigsqcup_D S = x$$

which means $s_2 \in S_x$. Thus, $s_2 \sqsubseteq u$, and so $s \sqsubseteq u$. This proves u is an upperbound for S. Then $x = \bigsqcup S \sqsubseteq u$. Since it is clear that x is an upperbound S_x, we have that $\bigsqcup S_x = x$. $\Box$

A subdomain V of a domain D carries with it two different notions of μ topology: Its *intrinsic* μ topology and its *inherited* μ topology. The *intrinsic* μ topology on V is the one it possesses as a domain in its own right. It is denoted as usual by μ_V. The *inherited* μ topology on V is the relative μ topology it receives as a subspace of D, and we denote it by $\mu_D|_V$.

Theorem 3.4.1 *The intrinsic μ topology on a subdomain is the same as its inherited μ topology.*

proof Let V be a subdomain of a domain D. First we show $\mu_V \subseteq \mu_D|_V$. The space (V, μ_V) has a basis consisting of sets of the form $\{t \in V : a \ll_V t \sqsubseteq x\}$, where $a, x \in V$. Because V is a subdomain and $a, x \in V$,

$$\{t \in V : a \ll_V t \sqsubseteq x\} = (\Uparrow_D a \cap \downarrow_D x) \cap V,$$

which shows that each basic open subset of (V, μ_V) is an open subset of $(V, \mu_D|_V)$. Hence, $\mu_V \subseteq \mu_D|_V$. For the inclusion $\mu_D|_V \subseteq \mu_V$, let U be an open subset of $(V, \mu_D|_V)$ and $x \in U$. First, since $U \in \mu_D|_V$, there is an open subset W of (D, μ_D) with $U = V \cap W$. Thus, there is $a \in W$ with $a \ll_D x$ and $\Uparrow_D a \cap \downarrow_D x \subseteq W$. Next, $\mathord{\downarrow\!\!\downarrow}_V x$ is a directed subset of D with

$\bigsqcup_D \mathord{\downarrow\!\!\downarrow}_V x = x \in W$, so by Lemma 3.4.1, W contains a directed subset S_x of $\mathord{\downarrow\!\!\downarrow}_V x$ whose supremum in D is x. By interpolation in D,

$$(\,\exists\, c \in S_x \subseteq \mathord{\downarrow\!\!\downarrow}_V x \subseteq V\,)\; a \ll_D c \ll_D x,$$

where we have $c \ll_D x$ because $\ll_V \subseteq \ll_D$ for the subdomain V. Then we claim that

$$x \in \mathord{\uparrow\!\!\uparrow}_V c \cap \mathord{\downarrow}_V x \subseteq V \cap W = U.$$

If $z \in V$ with $c \ll_V z \sqsubseteq x$, then $a \ll_D c \ll_D z \sqsubseteq x$, where $c \ll_D z$ holds once again because $\ll_V \subseteq \ll_D$. Then $z \in \mathord{\uparrow\!\!\uparrow}_D a \cap \mathord{\downarrow}_D x \subseteq W$. Then $z \in V \cap W = U$ which proves the claim. Thus, the set U is the union of open sets from (V, μ_V). This proves $\mu_D|_V \subseteq \mu_V$. $\square$

We consider a few examples which illustrate the delicate nature of the last result.

Example 3.4.3 Let $\mathbb{A}^\top := \mathbb{A} \cup \{\mathbb{N}\} \subset \mathcal{P}\omega$, where $\mathbb{A}$ is the antichain of Example 3.4.1. Then

(i) $\mathbb{A}^\top$ is a μ closed subset of $\mathcal{P}\omega$: As a sequence with its limit adjoined, it is even μ compact. This set is not μ open.

(ii) $(\mathbb{A}^\top, \subseteq)$ is a continuous dcpo.

(iii) $(\mathbb{A}^\top, \subseteq)$ is not a subdomain of $\mathcal{P}\omega$: All of its elements are compact, but none of these are compact in $\mathcal{P}\omega$.

(iv) The intrinsic and inherited μ topologies on $\mathbb{A}^\top$ do not agree. The intrinsic μ topology on $\mathbb{A}^\top$ is discrete, but its inherited μ topology is not. The set $\{\mathbb{N}\}$ is not open in the inherited μ topology because (A_n) is an infinite sequence of distinct elements of $\mathbb{A}^\top$ which converge to $\mathbb{N}$.

Thus, Theorem 3.4.1 need not hold if V is not a subdomain.

Example 3.4.4 Let $(D, \sqsubseteq)$ be a domain with an element x that is not compact. Let $V = \{x\}$. Then

(i) $(V, \sqsubseteq)$ is a continuous dcpo.

(ii) V is a μ closed subset of D.

(iii) The intrinsic and inherited μ topologies on V agree.

(iv) V is not a subdomain of D: The element x is compact in V but not in D.

Hence, a μ closed subset which is a continuous dcpo whose intrinsic and inherited μ topologies agree need not be a subdomain. That is to say, Theorem 3.4.1 has no obvious converse.

To say that a function $f : V \to E$ between a subdomain $V \subseteq D$ and another domain E is μ continuous refers *by definition* to the intrinsic μ topology on V. In light of Theorem 3.4.1, the advantage of subdomains is that their intrinsic and inherited μ topologies agree. We saw in Example 3.4.3 that this is not true in general, even for a μ closed subset $V \subseteq D$ which is a domain in its own right. The effect of this result on mappings like $f : V \to E$ is that *intrinsic continuity implies inherited continuity.* Later we will see an example that will give us a better feel for what a nice property this really is. For the time being, however, we have the following result.

Proposition 3.4.2 *Let U and V be subdomains of a domain D with $D = U \cup V$. Given μ continuous mappings between domains*

$$f : U \to E \text{ and } g : V \to E$$

which agree on $U \cap V$, the mapping

$$h : D \to E$$

given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in U; \\ g(x) & \text{if } x \in V, \end{cases}$$

is μ continuous.

proof Let $C \subseteq E$ be a μ closed subset of E. Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. We are going to prove that $f^{-1}(C)$ and $g^{-1}(C)$ are both μ closed subsets of D. By μ continuity of g, $g^{-1}(C)$ is a closed subset of (V, μ_V). By Theorem 3.4.1, $g^{-1}(C)$ is a closed subset of $(V, \mu_D|_V)$. Then by basic topology, there is a μ closed subset K of D such that $g^{-1}(C) = K \cap V$ But V is a μ closed subset of D according to Proposition 3.4.1, so $g^{-1}(C)$ is a μ closed subset of D. The very same argument applies to $f^{-1}(C)$. Since the inverse image of closed sets is closed, the mapping h is μ continuous. □

An important corollary then follows easily.

Corollary 3.4.1 *Let U and V be subdomains of a domain D with $D = U \cup V$. Given Scott continuous mappings between dcpo's*

$$f : U \to E \text{ and } g : V \to E,$$

which agree on $U \cap V$, the mapping

$$h : D \to E$$

given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in U; \\ g(x) & \text{if } x \in V, \end{cases}$$

is μ continuous.

proof By Proposition 3.2.1, the mappings f and g are μ continuous between domains because they are Scott continuous between domains. Now Proposition 3.4.2 applies. $\square$

It is worth pointing out that the maps f and g in the last result may not take μ open sets in E to μ *open sets in D.* Indeed, this happens iff the subdomains are μ open as well.

Secondly, we should emphasize that the pasting lemma's of this section are not the usual ones found in basic topology. We *do not* begin with continuous functions $f : U \to E$ and $g : V \to E$ defined on *subspaces.* Instead, we begin with functions defined on *subsets,* continuous with respect to an entirely different topology (the intrinsic μ topology), and then *prove* that these subsets are in fact closed subspaces. Indeed, this is what makes the technique useful.

Example 3.4.5 Let $\mathbb{A}^\top \subseteq \mathcal{P}\omega$ be the μ closed set of Example 3.4.3, which is not a subdomain, but is a continuous dcpo in its own right. Define a mapping

$$f : \mathbb{A}^\top \to \{0, 1\}$$

by

$$f(x) = \begin{cases} 0 & \text{if } x = A_n, n \geq 0; \\ 1 & \text{if } x = \mathbb{N}; \end{cases}$$

where the order on $\{0, 1\}$ is $x \sqsubseteq y \Leftrightarrow x = y$ or $(x = 0 \;\&\; y = 1)$.

As a function between domains, f is Scott continuous, and so continuous as a map from $\mathbb{A}^\top$ with its intrinsic μ topology to $\{0, 1\}$ with its μ topology.

However, this map does not reflect closed sets of $\{0,1\}$ onto closed sets of the larger domain $\mathcal{P}\omega$. Indeed,

$$f^{-1}(\{0\}) = \{A_n : n \geq 0\}.$$

That is, f is not continuous as a map from $\mathbb{A}^\top$ with its inherited μ topology into $\{0,1\}$ with its μ topology.

The next result gives a simple way to recognize subdomains.

Theorem 3.4.2 *For a subset V of a continuous dcpo D, the following are equivalent:*

(i) *V and $D \setminus V$ are both closed under directed suprema.*

(ii) *V is a subdomain of D and $D \setminus V$ is closed under directed suprema.*

(iii) *V and $D \setminus V$ are both subdomains of D.*

(iv) *V is both open and closed in the μ topology on D.*

proof (iv) $\Rightarrow$ (iii): By Lemma 3.1.1(iv), V is closed under directed suprema, which means that if $S \subseteq V$ is directed, then $\bigsqcup_D S \in V$. This implies that V is a dcpo. In addition, this also means that $(\forall x, y \in V)\, x \ll_D y \Rightarrow x \ll_V y$. We will use this to show that V is a continuous poset as follows. If $x \in V$, then since V is μ open, there is $a \ll_D x$ with $\Uparrow_D a \cap \downarrow_D x \subseteq V$. Consider the set

$$A_x := \{b \in D : a \ll_D b \ll_D x\} \subseteq V.$$

It is nonempty by interpolation in D, directed by the directedness of $\mathord{\downarrow\!\!\downarrow}_D x$, and since $\ll_D \cap V^2 \subseteq \ll_V$, A_x is a subset of $\mathord{\downarrow\!\!\downarrow}_V x \subseteq V$. It is equally clear that $\bigsqcup_D A_x = x$. Thus, $\mathord{\downarrow\!\!\downarrow}_V x$ contains a directed subset with supremum x, which implies that it too is directed with supremum x. This proves that V is a continuous poset and hence a domain.

Finally, we finish characterizing its approximation relation. Suppose that $x, y \in V$ with $x \ll_V y$. To prove $x \ll_D y$ it is enough to consider a directed subset $S \subseteq D$ with $y = \bigsqcup_D S$. By Lemma 3.4.1, V contains a directed subset S_y of S with supremum y. Since $x \ll_V y$, some $s \in S_y \subseteq S$ is above x. Thus, $x \ll_D y$. This shows that V is a subdomain; its complement $D \setminus V$ is also μ clopen, so it too is a subdomain. (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are obvious.

(i) $\Rightarrow$ (iv): First we show that V is μ closed. To prove that $D \setminus V$ is μ open, let $x \in D \setminus V$. First we claim that

$$(\exists\, a \in D)\, a \ll_D x \;\&\; \{b \in D : a \ll_D b \ll_D x\} \subseteq D \setminus V.$$

If not, for every $a \ll_D x$, there is $b \in V$ with $a \ll_D b \ll_D x$. The collection of all such b forms a directed subset of V with supremum x, which gives the contradiction $x \in V$. Thus, there is an $a \ll_D x$ with $\Uparrow_D a \cap \Downarrow_D x \subseteq D \setminus V$. Now let $t \in D$ be any element with $a \ll_D t \sqsubseteq x$. Then

$$\{z \in D : a \ll_D z \ll_D t\}$$

is a directed subset of $D \setminus V$ with supremum t. But $D \setminus V$ is closed under directed suprema, so $t \in D \setminus V$. Then we have $x \in \Uparrow_D a \cap \downarrow_D x \subseteq D \setminus V$. This proves $D \setminus V$ is μ open. Hence, V is μ closed. But now the very same argument shows that $D \setminus V$ is μ closed. Hence, V is μ open too. $\Box$

Recall that the μ topology on a domain is zero-dimensional. Because of this, it has a basis of the aforementioned *clopen* sets, that is, sets both closed and open in the μ topology.

Corollary 3.4.2 *For a subset V of a domain D, the following are equivalent:*

(i) *The set V is a μ open subdomain of D.*

(ii) *The set V is a subdomain of D with $\mu_V \subseteq \mu_D$.*

(iii) *The set V is a μ clopen subset of D.*

proof (iii) $\Rightarrow$ (ii) By Theorem 3.4.2, V is a subdomain. But then by Theorem 3.4.1, $\mu_V = \mu_D|_V$. Because V is a μ open subset of D, $\mu_D|_V \subseteq \mu_D$. (ii) $\Leftarrow$ (i) First, $V \in \mu_V \subseteq \mu_D$, so V is a μ open subset of D. (i) $\Rightarrow$ (iii) Proposition 3.4.1 shows that V is also μ closed. $\Box$

Corollary 3.4.3 *For a μ open subset V of a continuous dcpo D, the following are equivalent:*

(i) *V is closed under directed suprema in D.*

(ii) *V is a μ closed subset of D.*

(iii) *V is a subdomain of D.*

proof (iii) $\Rightarrow$ (ii): Proposition 3.4.1. (ii) $\Rightarrow$ (i): Lemma 3.1.1(iv).
(i) $\Rightarrow$ (iii): Then $D \setminus V$ is μ closed and so closed under directed suprema by Lemma 3.1.1(iv). If V is also closed under directed suprema, then Theorem 3.4.2 implies V is clopen and hence a subdomain. □

Here are some examples of μ clopen sets.

Lemma 3.4.2 *Let D be a continuous dcpo. Then*

(i) *Every Scott open subset of D is μ clopen.*

(ii) *Every Scott closed subset of D is μ clopen.*

(iii) *Every finite subset of $K(D)$ is μ clopen.*

(iv) *If $a \ll x$, then $\Uparrow a \cap \downarrow x$ is μ clopen.*

proof (i) A Scott open set is closed under directed suprema because it is an upper set. Its complement, a Scott closed set, is also closed under directed suprema. By Theorem 3.4.2, it must be μ clopen. (ii) Using (i), a Scott closed set is μ clopen because its complement is. (iii) Every compact element $x \in D$ gives rise to a μ open set $\{x\}$ by Lemma 3.1.1. This set is closed because the μ topology is Hausdorff (Theorem 3.1.1). (iv) This is the intersection of a Scott open set and a Scott closed set. Then it is the intersection of μ clopen sets according to (i) and (ii). □

With μ clopen sets, we can paste together any number of mappings defined on subdomains.

Proposition 3.4.3 *Let $\{V_i : i \in I\}$ be a collection of μ clopen subsets of a domain D with $D = \bigcup_{i\in I} V_i$. Given a collection*

$$\{f_i : V_i \to E : i \in I\}$$

of μ continuous mappings into the domain E with $f_i = f_j$ on $V_i \cap V_j$, the map

$$h : D \to E$$

given by

$$h(x) = f_i(x) \text{ for } x \in V_i$$

is μ continuous.

proof If $U \subseteq E$ is a μ open subset of the domain E, then

$$h^{-1}(U) = \bigcup_{i \in I} f_i^{-1}(U).$$

But $f_i^{-1}(U)$ is an open subset of (V_i, μ_{V_i}), so by Corollary 3.4.2, it must be a μ open subset of D. Then $h^{-1}(U)$ is a μ open subset of D. $\square$

The results of this section can be applied to provide us with a clear understanding of the difference between Scott continuity and μ continuity. Since we know that all Scott continuous maps are μ continuous, one can wonder when it is that all μ continuous maps are monotone (that is, Scott continuous). In addition, because the μ continuous maps are closed under pasting, one can ask when it is that the Scott continuous mappings have this property as well. As luck would have it, the answer to both of these questions is the same.

Proposition 3.4.4 *For a continuous dcpo D, the following are equivalent:*

(i) *Every μ continuous map $f : D \to D$ is monotone.*

(ii) *For any domain E, every μ continuous map $f : D \to E$ is monotone.*

(iii) *The μ topology on D is the same as the Scott topology.*

(iv) *The order $\sqsubseteq$ on D is discrete: $x \sqsubseteq y \Leftrightarrow x = y$.*

proof (iv) $\Rightarrow$ (iii): The order is discrete so every point of D is compact. By Lemma 3.1.1(v), $\{x\}$ is a μ open subset of D, for all $x \in D$. Hence the μ topology on D is discrete. However, using Lemma 3.1.1(iii), the upper set of a μ open set is Scott open. Then $\uparrow\{x\} = \{x\}$ is also Scott open, for all $x \in D$. This proves that both topologies in question are discrete.
(iii) $\Rightarrow$ (ii): If $U \subseteq E$ is a Scott open set, then it is μ open in E by Lemma 3.1.1(ii). Then $f^{-1}(U)$ is μ open in D and hence Scott open in D by assumption. This means that f is Scott continuous which proves that it is monotone.
(ii) $\Rightarrow$ (i): Take $E = D$.
(i) $\Rightarrow$ (iv): Let $x \sqsubseteq y$. If $y \sqsubseteq x$, then $x = y$, and we are done. Then assume $y \not\sqsubseteq x$. In this case, there must be a Scott open set U which contains y but not x. Define a mapping $f : D \to D$ by

$$f(t) = \begin{cases} x & \text{if } t \in U; \\ y & \text{otherwise.} \end{cases}$$

The set U and its complement are μ clopen by Lemma 3.4.2, so the function f is formed by pasting together two constant maps defined on the subdomains U and $D \setminus U$, respectively. By Proposition 3.4.2, f is μ continuous. Then by (i), f is monotone. Since $x \sqsubseteq y$, we have $h(x) = y \sqsubseteq h(y) = x$, which gives $x = y$. $\Box$

We will not quote the result about pasting for obvious aesthetical reasons. However, if pasting together Scott continuous maps defined on subdomains gives another Scott continuous map, then the argument given in (i) $\Rightarrow$ (iv) above shows that the order is trivial. The converse, that the order being trivial implies the Scott continuous maps are closed under pasting, is clear.

Corollary 3.4.4 *For a continuous dcpo D, the following are equivalent:*

(i) *Every μ continuous map $f : D \to D$ has a fixed point.*

(ii) *The domain D is a one point domain.*

proof (i) $\Rightarrow$ (ii): If the domain has two points $x \neq y$, the argument given in (i) $\Rightarrow$ (iv) of Proposition 3.4.4 gives a μ continuous map without a fixed point. The other direction is trivial. $\Box$

Of course, no disconnected space has the fixed point property, so the last corollary is unsurprising. However, it seems to the author that the space of selfmaps of a disconnected space does not usually possess a subset capable of acting as a nondegenerate lambda model. As we know, a monotone mapping is Scott continuous iff it is μ continuous. Then if the property of monotonicity could be written purely in terms of the μ topology, one would obtain an analogue of "Scott continuous map" for zero-dimensional T_1 spaces in general. In the process, one may even obtain a strong candidate for a cartesian closed category of zero-dimensional T_1 spaces with nondegenerate lambda models. For a nice discussion of the interesting problem of lambda models, see [17].

3.5 Ideal Maps

We consider an application of the μ topology to ideal maps.

Definition 3.5.1 A splitting $s : D \to D$ on a dcpo D is *ideal* if for all sequences (a_n) in D with $s(a_n) \sqsubseteq a_{n+1}$ for all n, we have $s(\bigsqcup a_n) = \bigsqcup s(a_n)$.

Lemma 3.5.1 *Let $s : D \to D$ be a splitting on a dcpo D. Then*

(i) *The map s is ideal iff for all sequences (a_n) with $s(a_n) \sqsubseteq a_{n+1}$, we have $s(\bigsqcup a_n) = \bigsqcup a_n$.*

(ii) *If s is ideal, then $\bigsqcup s^n(x) \in \text{fix}(s)$, for all $x \in D$. In addition, $\text{fix}(s)$ is closed under suprema of increasing sequences.*

Ideal mappings form a class of splittings with the fixed point property closed under composition.

Proposition 3.5.1 *Let D be a dcpo.*

(i) *If $s, r : D \to D$ are ideal, then $s \circ r : D \to D$ is ideal.*

(ii) *If $s : D \to D$ is ideal and $I \subseteq D$ is closed under directed suprema with $s(I) \subseteq I$, then the restriction of s to I is an ideal map on I.*

proof (i) If (a_n) is a sequence with $s(r(a_n)) \sqsubseteq a_{n+1}$, $r(a_n) \sqsubseteq a_{n+1}$, since s is a splitting. Hence, $r(\bigsqcup a_n) = \bigsqcup r(a_n)$. But now observe that

$$r(a_n) \sqsubseteq s(r(a_n)) \sqsubseteq a_{n+1} \sqsubseteq r(a_{n+1}),$$

so $s(\bigsqcup r(a_n)) = \bigsqcup s(r(a_n))$, as s is ideal. Finally,

$$s(r(\bigsqcup a_n)) = s(\bigsqcup r(a_n)) = \bigsqcup s(r(a_n)) = \bigsqcup a_n,$$

which proves $s \circ r$ is ideal. The proof of (ii) is easy. $\Box$

Now we need techniques for proving that a mapping is ideal.

Proposition 3.5.2 *Let D be a dcpo.*

(i) *A Scott continuous splitting is ideal.*

(ii) *If $f : D \to D$ is Scott continuous, then its restriction to the dcpo $I(f)$ is an ideal map on $I(f)$.*

The previous result generalizes considerably on a domain.

Proposition 3.5.3 *A μ-σ continuous splitting on a domain is ideal.*

proof Let $s : D \to D$ be a splitting which is μ-σ continuous. Given a sequence (a_n) with $a_n \sqsubseteq s(a_n) \sqsubseteq a_{n+1}$, we always have

$$\bigsqcup a_n = \bigsqcup s(a_n) \sqsubseteq s(\bigsqcup a_n).$$

By μ-σ continuity, $s(\bigsqcup a_n) \sqsubseteq \bigsqcup s(a_n)$, according to Lemma 3.2.3. $\Box$

The next results are some of the most useful methods for recognizing an ideal map.

Proposition 3.5.4 *Let D be a domain with a measurement $\mu \to \sigma_D$. If $s : D \to D$ is a splitting with μ-σ continuous measure, then s is ideal.*

proof We gave the proof of this in Proposition 3.3.3. $\Box$

In the next result, recall that any measurement $\mu \to \sigma_D$ is strictly monotone.

Proposition 3.5.5 *Let $\mu : D \to [0, \infty)^*$ be a strictly monotone map on a dcpo D. If $s : D \to D$ is a splitting for which there exists $r < 1$ such that*

$$\mu s \leq r \cdot \mu,$$

then s is ideal.

proof It is easy to see that any element in $\ker \mu$ is maximal. If (a_n) is a sequence with $a_n \sqsubseteq s(a_n) \sqsubseteq a_{n+1}$, then $(\forall n > 0)\ \mu a_{n+1} \leq r^n \mu a_1$. Thus,

$$\mu(\bigsqcup a_n) \leq \mu a_{n+1} \leq r^n \mu a_1,$$

for all $n > 0$. Then $\bigsqcup a_n$ is maximal in D and so must be a fixed point. This proves s is ideal by Lemma 3.5.1(i). $\Box$

Thus, the μ topology via measurement enables us to recognize ideal maps. This technique is easier to appreciate when one realizes just how uncommon ideal mappings really are.

Lemma 3.5.2 *For a dcpo D, the following are equivalent:*

(i) *Every splitting on D is ideal.*

(ii) *The supremum of every strictly increasing sequence in D is maximal.*

proof (i) $\Rightarrow$ (ii): Let (a_n) be an increasing sequence with $a_n \neq a_{n+1}$ for all n. Write $a = \bigsqcup a_n$ and, using the fact that D is a dcpo, choose a maximal element $m \in \max D$ with $a \sqsubseteq m$. Define a splitting

$$s : D \to D$$

$$s(x) = \begin{cases} x_{n+1} & \text{if } x = x_n; \\ m & \text{if } x = a; \\ x & \text{otherwise.} \end{cases}$$

Then $\bigsqcup x_n = \bigsqcup s^n(x_1) = a$. However, the splitting s is ideal, so $a \in \text{fix}(s)$. This gives $s(a) = a = m$ which proves that $\bigsqcup x_n$ is maximal. (ii) $\Rightarrow$ (i): Let $s : D \to D$ be a splitting and $s(a_n) \sqsubseteq a_{n+1}$. If this sequence is eventually constant, its supremum is a fixed point of s, and so obviously one preserved by s. On the other hand, if it is not eventually constant, its supremum must be maximal by (ii). But

$$\bigsqcup_{n\geq 0} a_n \sqsubseteq s(\bigsqcup_{n\geq 0} a_n) = \bigsqcup_{n\geq 0} s(a_n),$$

so the inequality on the left is actually an equality. □

Corollary 3.5.1 *For an ω-continuous domain D, the following are equivalent:*

(i) *Every splitting on D is μ-σ continuous.*

(ii) *The domain D is algebraic with $D \setminus \max D \subseteq K(D)$.*

(iii) *Every splitting on D is ideal.*

proof (iii) $\Rightarrow$ (ii): If $Letx \in D \setminus \max\ D$ and take a sequence (x_n) in $\twoheaddownarrow x$ with supremum x. Were it not eventually constant, its supremum would have to be maximal by Lemma 3.5.2, contradicting $x \notin \max\ D$. Hence, $(\exists n)\, x_n = x \ll x$. Then every element off the top is compact, which makes it clear that D is algebraic. (ii) $\Rightarrow$ (i): All splittings are μ hence μ-σ continuous at compact elements by Lemma 3.2.2. Any element of D which is not compact is maximal and hence a fixed point of any splitting. But splittings are μ-σ continuous at fixed points as well, according to Lemma 3.2.4. (i) $\Rightarrow$ (iii): Proposition 3.5.3. □

Now we turn to the question of *why* one would be interested in ideal maps to begin with.

Definition 3.5.2 An *inductive property* on a dcpo D is a subset $P \subseteq D$ closed under directed suprema together with two ideal maps $l : D \to D$ and $r : D \to D$ such that

$$x \in P \Rightarrow lx \in P \text{ or } rx \in P$$

for all $x \in D$. We can write an inductive property as a triple (P, l, r).

The complement of an inductive property is a deductive property.

Definition 3.5.3 A *deductive property* on a dcpo D is a subset $P \subseteq D$ together with a pair of ideal maps $l : D \to D$ and $r : D \to D$ such that the triple $(D \backslash P, l, r)$ forms an inductive property.

Theorem 3.5.1 (Induction) *If P is an inductive property on a dcpo D, then*

$$x \in P \Rightarrow P \cap \uparrow x \cap (\mathrm{fix}(l) \cup \mathrm{fix}(r)) \neq \emptyset$$

for all $x \in D$.

proof Define a splitting $s : P \to P$ by

$$s(x) = \begin{cases} l(x) & \text{if } l(x) \in P; \\ r(x) & \text{otherwise.} \end{cases}$$

Let $(a_n)_{n \geq 0}$ be a sequence in P with $a_n \sqsubseteq s(a_n) \sqsubseteq a_{n+1}$ for all n. Then there is an infinite subsequence of (a_n) named (b_i), which has the same supremum as (a_n), and for which we also have that either

$$(\forall i)\, l(b_i) \sqsubseteq b_{i+1} \;\&\; l(b_i) \in P$$

or

$$(\forall i)\, r(b_i) \sqsubseteq b_{i+1} \;\&\; l(b_i) \notin P.$$

In the first case, the idealness of l, combined with the fact that P is a dcpo, gives

$$l(\bigsqcup a_n) = l(\bigsqcup b_i) = \bigsqcup l(b_i) = \bigsqcup b_i = \bigsqcup a_n \in P.$$

Thus, $a_0 \sqsubseteq l(\bigsqcup a_n) = \bigsqcup a_n$. In the second case, we must have $r(b_i) \in P$, and so the same argument gives $a_0 \sqsubseteq r(\bigsqcup a_n) = \bigsqcup a_n$. Finally, given a point $x \in P$, we set $a_n = s^n x$, for $n \geq 0$, and see that

$$\bigsqcup s^n(x) \in P \cap \uparrow x \cap (\mathrm{fix}(l) \cup \mathrm{fix}(r)),$$

which finishes the proof. $\Box$

Corollary 3.5.2 (Deduction) *If P is a deductive property on a dcpo D, then*

$$\uparrow x \cap (\text{fix}(l) \cup \text{fix}(r)) \subseteq P \Rightarrow x \in P,$$

for all $x \in D$.

It is interesting that induction on the naturals has the form of Theorem 3.5.1.

Example 3.5.1 Let $p : \mathbb{N} \to \{\bot, \top\}$ be a function. The set

$$P = \{n \in \mathbb{N} \cup \{\infty\} : (\forall k < n)\, p(k) = \top\}$$

is closed under directed suprema in $\mathbb{N} \cup \{\infty\}$. The successor function

$$\text{succ}\, n = \begin{cases} n+1 & \text{if } n \in \mathbb{N} \\ \infty & \text{if } n = \infty \end{cases}$$

is ideal. If p has the property that for all $n \in \mathbb{N}$,

$$p(n) = \top \Rightarrow p(n+1) = \top,$$

then $(P, \text{succ}, \text{succ})$ is an inductive property on $\mathbb{N} \cup \{\infty\}$. In this case, Theorem 3.5.1 says that $p(0) = \top \Rightarrow p(n) = \top$ for all $n \in \mathbb{N}$.

The connectedness of $\mathbb{R}$ may now be proven by induction.

Example 3.5.2 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map on the real line. The set

$$C(f) = \{[a, b] \in \mathbf{I}\mathbb{R} : f(a) \cdot f(b) \leq 0\}$$

is μ closed and so it is closed under directed suprema. The mappings

$$\text{left} : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R} \text{ and right} : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$$

are ideal, using either Proposition 3.5.4 or Proposition 3.5.5, and

$$x \in C(f) \Rightarrow \text{left}\, x \in C(f) \text{ or right}\, x \in C(f),$$

for all $x \in \mathbf{I}\mathbb{R}$. Thus, $(C(f), \text{left}, \text{right})$ is an inductive property on $\mathbf{I}\mathbb{R}$. By induction, if f changes sign on $[a, b]$, it must have at least one zero on $[a, b]$.

The compactness of $[0, 1]$ can be proven by deduction (assuming the completeness of $\mathbb{R}$).

Example 3.5.3 Let $\{U_\alpha\}$ be an open cover of $[0,1]$. Consider the set

$$P = \{x \in \mathbf{I}\mathbb{R} : x \textit{ can be finitely covered by } \{U_\alpha\}\}.$$

First, P is a μ open subset of $\mathbf{I}\mathbb{R}$. If $[a,b] \in P$, write $[a,b] \subseteq \bigcup_{i=1}^{n} U_i = V$. Because V is an open subset of $\mathbb{R}$,

$$(\,\exists\, \varepsilon > 0\,)\; a \in (a-\varepsilon, a+\varepsilon) \subseteq V \;\&\; b \in (b-\varepsilon, b+\varepsilon) \subseteq V,$$

which means $\Uparrow[a-\varepsilon, b+\varepsilon] \cap \downarrow[a,b] \subseteq P$. Then P is μ open and so $\mathbf{I}\mathbb{R} \setminus P$ is closed under directed suprema. Now observe that for all $x \in \mathbf{I}\mathbb{R}$,

$$\text{left}\, x \in P \;\&\; \text{right}\, x \in P \Rightarrow x \in P.$$

Thus, $(P, \text{left}, \text{right})$ is a deductive property. Finally,

$$\uparrow[0,1] \cap (\text{fix(left)} \cup \text{fix(right)}) = \{[t] : t \in [0,1]\} \subseteq P,$$

and so by deduction $[0,1] \in P$.

Notice that the μ topology in the previous applications completely describes the inductive property (P, l, r): Not only is it employed to show that l and r are ideal, but, in addition, the set P is either μ open or μ closed. This is not rare at all.

Another application of the induction principle is to program correctness. We derive the following procedure for establishing the correctness of an algorithm:

(i) For an algorithm a, let $\Diamond a$ be its domain, the set of all *possible* inputs, and let $\Box a := \{x \in \Diamond a : a \text{ works correctly on } x\}$. Trivially, then, $\Box a \subseteq \Diamond a$.

(ii) Show that $\Box a$ is a deductive property over $\Diamond a$.

(iii) Use deduction to show $\Diamond a \subseteq \Box a$.

(iv) Conclude that a works correctly on all inputs since $\Box a = \Diamond a$.

We now apply this idea to list processing algorithms on $[S]$. Notice that it has discrete μ topology. Hence, all subsets of $[S]^2$ and $[S]$ are μ clopen, and all splittings on these spaces are μ continuous hence ideal.

Example 3.5.4 Consider the the following ML program to merge two sorted lists of integers:

$$
\begin{array}{ll}
\text{fun merge}(\ [\,], ys\ ) & = ys\ :\ \text{int list} \\
\quad |\ \text{merge}(\ xs, [\,]\ ) & = xs \\
\quad |\ \text{merge}(\ x :: xs, y :: ys\ ) & = \text{if } x \leq y \text{ then} \\
 & \quad\ x :: \text{merge}(\ xs, y :: ys\ ) \\
 & \ \ \text{else} \\
 & \quad\ y :: \text{merge}(\ x :: xs, ys\ );
\end{array}
$$

(1) *Identify the domain of the algorithm and define correctness.*

$$
\begin{array}{l}
\Diamond\text{merge} := \{\ (x, y) \in [\text{int}]^2 : x \text{ sorted}, y \text{ sorted}\ \} \\
\Box\text{merge} := \{\ (x, y) \in \Diamond\text{merge} :\ \text{merge}(x, y) \text{ sorted}\ \}
\end{array}
$$

(2) *Show that* $\Box$merge *is a deductive property on* $\Diamond$merge.

Consider the splitting

$$\pi : \Diamond\text{merge} \to \Diamond\text{merge}$$

defined by

$$
\begin{array}{ll}
\pi(\ [\,], ys\ ) & = (\ [\,], ys\ ) \\
\pi(\ xs, [\,]\ ) & = (\ xs, [\,]\ ) \\
\pi(\ x :: xs, y :: ys\ ) & = (\ xs, y :: ys\ ) \text{ if } x \leq y \\
 & = (\ x :: xs, ys\ ) \text{ otherwise.}
\end{array}
$$

If $(x, y) \in \Diamond$merge, then

$$\pi(x, y) \in \Box\text{merge} \Rightarrow (x, y) \in \Box\text{merge}.$$

Thus, ($\Box$merge, π, π) is a deductive property over $\Diamond$merge.

(3) *Use deduction to show that* $\Diamond$merge $\subseteq \Box$merge.

The fixed points of π are $\text{fix}(\pi) = \{\ ([\,], y) : y \text{ sorted}\ \} \cup \{\ (x, [\,]) : x \text{ sorted}\ \}$.

Given $(x, y) \in \Diamond\text{merge}$,

$$\uparrow(x, y) \cap \text{fix}(\pi) \subseteq \text{fix}(\pi) \subseteq \Box\text{merge}.$$

Then, by deduction, $(x, y) \in \Box\text{merge}$.

(4) Consequently, merge is a correct algorithm.

Example 3.5.5 Consider the ML implementation of mergesort for lists of integers:

```
fun sort [ ] = [ ]
  | sort [x]= [x]
  | sort xs = let val n = length xs div 2
              in merge( sort( take(n,xs) ),
                        sort( drop(n,xs) ) )
              end;
```

The domain of sort is $\Diamond\text{sort} = [\text{int}]$ and $\Box\text{sort} := \{\ x \in [\text{int}] : \text{sort}(x) \text{ sorted }\}$.

Now consider the splittings

left : [int] $\to$ [int]	right : [int] $\to$ [int]
left xs = take(length xs div 2, xs)	right xs = drop(length xs div 2, xs)

By the correctness of merge,

($\Box$sort, left, right) is a deductive property over [int].

Finally,

$$\text{fix(left)} = \{[\ ]\} \subseteq \Box\text{sort}$$

and

$$\text{fix(right)} = \{[x] : x \in \text{int}\} \cup \{[\ ]\} \subseteq \Box\text{sort},$$

which by deduction proves that $\Box\text{sort} = [\text{int}]$, i.e., sort is correct.

The idea that the "recursive part" of an algorithm is somehow described by a collection of splittings is a good one to keep in mind. It will play a fundamental role in the work of the next chapter.

3.6 Summary

We have seen that all measurements $\mu \to \sigma_D$ have something in common: They yield a fixed topology on D that we have named the μ topology. In this way, we can now think of measurements as being a kind of metric for the μ topology on a domain. However, just as in traditional mathematics, where it definitely matters which metric one uses on a space, it matters here on domains which measurement one uses. This is best illustrated by the fixed point theorems that make use of the measure of a splitting: One cannot observe that the measure of a function is, say, μ-σ continuous, unless they are using the "correct" measurement.

This is one reason we chose to introduce the μ topology *after* measurement. Another reason is, of course, that one wants the μ topology to be understandable. It would look rather bizarre were one to study the μ topology independent of measurement: This would be like studying abstract point sets in a world where no one had any knowledge of metrics.

One final remark on this analogy "measurements are metrics for the μ topology." In classical mathematics, Urysohn's theorem tells us that every regular second countable space is metrizable. The analogous result holds for domains as follows. If the μ topology on D is separable, we know that its Scott topology has a countable basis $\{U_n : n \in \mathbb{N}\}$. This yields an embedding which induces the Scott topology everywhere

$$e : D \to \mathcal{P}\omega$$

$$e(x) = \{n \in \mathbb{N} : x \in U_n\}.$$

Now we compose this map with $|\cdot| : \mathcal{P}\omega \to [0, \infty)^*$ and the proof is finished.

3.7 Questions

(i) Continuous dcpo's may be viewed as zero-dimensional Hausdorff spaces in their μ topology. Which ones?

(ii) Given a domain D, is it possible to construct a new domain D^1, so that all μ continuous mapppings on D can be realized as Scott continuous mappings on D^1?

(iii) Are there domains D with measurements λ for which $[D \to D](\lambda)$ is maximal? That is, maximal in the sense that, given any $\mu : D \to E$ with $\mu \to \sigma_D$, we always have $[D \to D](\mu) \subseteq [D \to D](\lambda)$. Good cases to think of are $([S], \text{len})$, $(\Sigma^\infty, \frac{1}{2^{|\cdot|}})$ and $(\mathbf{I}\mathbb{R}, \mu)$.

(iv) In relation to (iii), why is $\mu[a, b] = b - a$ the "best" measurement on $\mathbf{I}\mathbb{R}$? Similarly for the other examples, and which domains in general admit a "best" measurement?

(v) The Scott topology on a domain is locally compact and sober. The Lawson topology on a coherent domain is compact Hausdorff. What special properties does the μ topology on a domain have? What about when the domain is coherent? Observe that the μ topology on Σ^∞ and on $[S]$ is locally compact. This hardly ever happens. When is the μ topology locally compact?

(vi) Is the μ topology on $\mathcal{P}\omega$ normal? Is it Lindelof? Is it second countable?

(vii) Which first countable domains admit a measurement $\mu \to \sigma_D$?

(viii) If $f : D \to D$ is a μ-σ map with monotone measure, is $I(f)$ necessarily a dcpo? Must it be μ closed? (Note: The last example of 3.2 shows that such a map can have no fixed points.)

(ix) What is the largest class of μ-σ maps closed under composition? What is the largest class of μ-σ maps with monotone measure closed under composition?

(x) When does a μ continuous map preserve improvements? If a μ continuous map preserves maximal elements, when is its restriction to $\max D$ Scott continuous?

(xi) Characterize lower sets and monotone mappings strictly in terms of the μ topology. Using the Scott topology we know that the order $\sqsubseteq$ on a domain D is given by

$$x \sqsubseteq y \Leftrightarrow (\forall U \in \sigma_D)\, x \in U \Rightarrow y \in U.$$

Can we write this in terms of the μ topology?

Chapter 4

Algorithms and Mappings Between Domains

In this chapter we consider algorithms as mappings between domains and seek to develop a reasonable and especially a simple notion of exactly what kinds of functions they are.

4.1 Algorithms

Algorithm is a simple idea, one easily explained to entering freshman in college. As such, its mathematical description ought to be equally accessible. This is our present goal.

An algorithm is a function *plus a definition* that specifies *how* its values are to be found. Probably the clearest and most concrete attempt at capturing the notion of algorithm is the class of partial recursive functions.

Definition 4.1.1 Let $\mathbb{N}_\perp$ denote the set $\mathbb{N} \cup \{\perp\}$, where $\perp$ is an element that does not belong to $\mathbb{N}$.

For instance, one could take $\perp = \{\mathbb{N}\}$, should the need arise.

Definition 4.1.2 A *partial function* on the naturals is a function

$$f : \mathbb{N}^n \to \mathbb{N}_\perp,$$

where $n \geq 1$. We say that f is *undefined* at x exactly when $f(x) = \perp$.

Thinking of f as an algorithm, $f(x) = \bot$ means that the program f crashed when we sent it input x.

Definition 4.1.3 The *composition* of a partial map $f : \mathbb{N}^n \to \mathbb{N}_\bot$ with partial mappings $g_i : \mathbb{N}^k \to \mathbb{N}_\bot$, $1 \leq i \leq n$, is the partial map

$$f(g_1, \cdots, g_n) : \mathbb{N}^k \to \mathbb{N}_\bot$$

$$f(g_1, \cdots, g_n)(x) = \begin{cases} f(g_1(x), \cdots, g_n(x)) & \text{if } (\forall i)\ g_i(x) \neq \bot; \\ \bot & \text{otherwise.} \end{cases}$$

That is, if in the process of trying to run the program f, the computation of one of its inputs fails, then the entire computation fails.

Definition 4.1.4 A partial map $f : \mathbb{N}^{n+1} \to \mathbb{N}_\bot$ is defined by *primitive recursion* from $g : \mathbb{N}^n \to \mathbb{N}_\bot$ and $h : \mathbb{N}^{n+2} \to \mathbb{N}_\bot$ if

$$f(\bar{x}, y) = \begin{cases} g(\bar{x}) & \text{if } y = 0; \\ h(\bar{x}, y-1, f(\bar{x}, y-1)) & \text{otherwise.} \end{cases}$$

where we have written $\bar{x} \in \mathbb{N}^n$.

This computationally is a counting loop. Notice that the rule of composition given above insists that in order for $f(\bar{x}, y)$ to be defined, f must be defined at $(\bar{x}, z)$, for all $z < y$.

Definition 4.1.5 The *minimization* of a partial function $f : \mathbb{N}^{n+1} \to \mathbb{N}_\bot$ is the partial function

$$\mu f : \mathbb{N}^n \to \mathbb{N}_\bot$$

$$\mu f(x) = \min\{y \in \mathbb{N} : (\forall z < y)\ f(x, z) \neq \bot \ \&\ f(x, y) = 0\}$$

with the convention that $\mu f(x) = \bot$ if no such y exists.

The intent of minimization is that the least y such that $f(x, y) = 0$ is calculable according to the natural algorithm: We begin iterating at 0, if $f(x, 0) = 0$, this is the answer. If $f(x, 0) = \bot$, the program has crashed, so the output is $\bot$. Otherwise, we proceed to $f(x, 1)$, and so on, in the same fashion. Hence the requirement that $f(x, z) \neq \bot$, for all $z < y$: Without it, we have no assurance that y was found using the algorithm above.

Definition 4.1.6 The class of *partial recursive functions* on the naturals is the smallest collection of partial maps $f : \mathbb{N}^n \to \mathbb{N}_\perp$ which contains the zero function, the successor, the projections, and is closed under composition, primitive recursion, and minimization.

The assertion that the partial recursives capture the notion of intuitively computable function on the naturals is known as the Church-Turing thesis. It may or may not be true. However, we can explain why they capture the sequential algorithms executable on machines today.

Imagine that you have just written a program in your favorite language. You then compile it, and the compiler translates it to assembly code. (From there another program translates the assembly code to machine code so it can be executed on a machine.) Assembly is a language where one programs with integers using very simple instructions. The programs one can write in assembly language can all be coded in the the following language:

$$
\begin{array}{rcl}
\text{stmt} & ::= & \textbf{clear}\, x \\
& | & \textbf{inc}\, x \\
& | & \textbf{dec}\, x \\
& | & \textbf{while}\, x \neq 0 \ \textbf{do} \ \text{stmt} \\
& | & \text{stmt}; \text{stmt}.
\end{array}
$$

where x can be any element taken from a countable set of variables. But the programs one may write in this language are exactly the partial recursive functions (one must explain a little bit here about how such a program may be regarded a function). Thus, every sequential algorithm on a machine is translated to an assembly language program which is the coding of a partial recursive function on the naturals.

For this reason, we naturally want our theory to allow a description of the partial recursive functions. At the same time, we dont want to view list algorithms as partial recursive functions on the naturals: We want to view them as functions on the domain of lists $[S]$. This is where domain theory helps a lot: Each domain can be thought of as representing a different data type. The nice part about a domain, however, is that it also has enough structure to allow us to solve *renee equations,* as we shall see shortly.

Hiding somewhere in the above remarks is a result due to Kleene. We mentioned earlier that primitive recursion amounts to the ability to program with counting loops. Minimization, on the other hand, is much more powerful: It represents the ability to program with while loops. It is clear, however, that all counting loops may be rewritten as while loops. Thus, primitive recursion is essentially redundant in the definition of partial recursive. Kleene confirmed this fact in 1936, provided one carries a few more initial functions.

Theorem 4.1.1 (Kleene) *The class of partial recursive functions on the naturals is the smallest collection of partial maps $f : \mathbb{N}^n \to \mathbb{N}_\perp$ which contains addition $+$, multiplication $\times$, the projection mappings, the predicate*

$$\leq (a,b) = \begin{cases} 1 & \text{if } a \leq b; \\ 0 & \text{otherwise,} \end{cases}$$

and is closed under composition and minimization.

We should point out here that we have used a different but equivalent set of initial functions.

4.2 Iterative Operations

We now begin our investigation of algorithms as maps between domains.

Definition 4.2.1 Let $(X, +)$ be a Hausdorff space with a binary operation that is associative. If (x_n) is a sequence in X, then its *infinite sum* is

$$\sum_{n\geq 1} x_n := \lim_{n\to\infty} (x_1 + \cdots + x_n)$$

provided that the limit of the partial sums on the right exists.

Definition 4.2.2 Let $+ : D^2 \to D$ be a binary operation on a domain. A point $x \in D$ is *idle* if there is a μ open set $\sigma(x)$ around x such that

(i) $(\sigma(x), +)$ is a semigroup, and

(ii) If (x_n) is any sequence in $\sigma(x)$ which converges to x in the μ topology, then
$$\sum_{n\geq 1} x_n \text{ exists and } \lim_{n\to\infty} \sum_{k\geq n} x_k = \lim_{n\to\infty} x_n.$$

The operation $+$ is said to be *idle* at x.

An idle point is one where the "unwinding" of a recursive definition stops. For example, $0 \in \mathbb{N}$, or the empty list.

Definition 4.2.3 Let D be a continuous dcpo. A μ continuous operation $+ : D^2 \to D$ is *iterative* if it has at least one idle point.

The definition of infinite sum extends to *any* sequence with an idle limit.

Definition 4.2.4 Let $(D, +)$ be a domain with an iterative operation. If (x_n) is a convergent sequence whose μ limit x is idle, we define its *suspended sum* to be

$$\overleftarrow{\sum_{n\geq 1}} x_n := x_1 + (x_2 + (\cdots + (x_{k-1} + \sum_{n\geq k} x_n)\cdots))$$

where $k \geq 1$ is the least integer such that $x_n \in \sigma(x)$ for all $n \geq k$.

Now we consider properties of iterative operations. The first is a technical point worth emphasizing.

Lemma 4.2.1 *The μ topology on a product of domains $D \times E$ is the product of the μ topologies on D and E*

proof If D and E are domains and $(x, y), (a, b) \in D \times E$, the equality

$$\Uparrow(x, y) \cap \downarrow(a, b) = (\Uparrow x \times \Uparrow y) \cap (\downarrow a \times \downarrow b) = (\Uparrow x \cap \downarrow a) \times (\Uparrow y \cap \downarrow b)$$

shows that the μ topology on the domain $D \times E$ is just the usual product topology on $(D, \mu_D) \times (E, \mu_E)$. $\Box$

Lemma 4.2.1 has an important consequence for iterative operations that we will often invoke by simply referring to the "μ continuity of $+$."

Lemma 4.2.2 *Let $(D, +)$ be a domain with an iterative operation. If (x_n) and (y_n) are sequences with $x_n \to x$ and $y_n \to y$ in the μ topology on D, then*

$$x_n + y_n \to x + y$$

in the μ topology on D.

Suspended sums have the same properties that infinite sums do.

Proposition 4.2.1 *Let $(D, +)$ be a domain with an iterative operation and (x_n) be a sequence with an idle limit. Then*

$$\overleftarrow{\sum_{n\geq 1}} x_n = x_1 + \overleftarrow{\sum_{n\geq 2}} x_n$$

and

$$\lim_{n\to\infty} \overleftarrow{\sum_{k\geq n}} x_k = \lim_{n\to\infty} x_n.$$

proof Let $x_n \to x$ and $k \geq 1$ be the least integer such that $x_n \in \sigma(x)$ for all $n \geq k$. Then the suspended sum of the sequence $\{x_n : n \geq i\}$, for $i \geq 1$, is

$$\overleftarrow{\sum_{n\geq i}} x_n := \sum_{n\geq i} x_n$$

for $i \geq k$, and

$$\overleftarrow{\sum_{n\geq i}} x_n := x_i + \overleftarrow{\sum_{n\geq i+1}} x_n,$$

for $1 \leq i \leq k-1$. First we prove the associativity. For $k > 1$, it is clear by the definition above. Then suppose that $k = 1$. In this case, all suspended sums are just infinite sums on the semigroup $(\sigma(x), +)$. Let (s_n) be the sequence of partial sums associated with the sum of the sequence $\{x_n : n \geq 1\}$:

$$\begin{aligned} s_1 &= x_1 \\ s_{n+1} &= s_n + x_{n+1}, \quad n \geq 1. \end{aligned}$$

Similarly, letting $t_2 = x_2$ and $t_{n+1} = t_n + x_{n+1}$ for $n \geq 2$, gives the sequence (t_n) of partial sums corresponding to the sum of the elements $\{x_n : n \geq 2\}$. By associativity and induction, we see that $x_1 + t_n = s_n$ for all $n \geq 2$. Finally, we compute as follows

$$\begin{aligned} \sum_{n\geq 1} x_n = \lim_{n\to\infty} s_n &= \lim_{n\to\infty} (x_1 + t_n) \\ &= \lim_{n\to\infty} x_1 + \lim_{n\to\infty} t_n \\ &= x_1 + \sum_{n\geq 2} x_n, \end{aligned}$$

using the μ continuity of $+$. For the other property,

$$\lim_{n\to\infty} \overleftarrow{\sum_{k\geq n}} x_k = \lim_{n\to\infty} \sum_{k\geq n} x_k = \lim_{n\to\infty} x_n,$$

since the sequence of suspended sums is eventually a sequence of infinite sums. □

Proposition 4.2.2 *Let $(D, +)$ be a domain with an iterative operation.*

(i) *If $+$ is idle at x, then $x + x = x$.*

(ii) *A compact element $x \in D$ is idle iff $x + x = x$.*

(iii) *If $+$ is idle at x and (x_n) is a sequence in $\sigma(x)$ which converges to x in the μ topology, then*

$$\sum_{n\geq 1} x_n + x = \sum_{n\geq 1} x_n.$$

proof (i) First, the sequence $x_n = x$ converges to x, which means that

$$s_n = \sum_{k\geq n} x_k$$

exists for all $n \geq 1$. However, (s_n) is a constant sequence with limit x. Thus, $s_n = s_{n+1} = x$, for all $n \geq 1$. By the associativity in Proposition 4.2.1,

$$x + x = x + \sum_{n\geq 2} x_n = \sum_{n\geq 1} x_n = x.$$

(ii) If x is compact, then $\{x\}$ is a μ open set around x. For (iii), let (s_n) be the sequence of partial sums associated with $\sum_{n\geq 1} x_n$. Then

$$\begin{aligned}
\sum_{n\geq 1} x_n + x &= \lim_{n\to\infty} s_n + \lim_{n\to\infty} x_n \\
&= \lim_{n\to\infty} s_n + \lim_{n\to\infty} x_{n+1} \\
&= \lim_{n\to\infty} (s_n + x_{n+1}) \\
&= \lim_{n\to\infty} s_{n+1} \\
&= \sum_{n\geq 1} x_n,
\end{aligned}$$

where we have used the μ continuity of $+$. $\square$

Here are a few simple examples of iterative operations.

Example 4.2.1 Data types.

(i) $([S], \cdot)$ concatenation of lists.

(ii) $(\mathbb{N}^*, +)$ addition of natural numbers.

(iii) $(\mathbb{N}^*, \times)$ multiplication of natural numbers.

(iv) $(\{\bot, \top\}, \vee)$ boolean or.

(v) $(\{\bot, \top\}, \wedge)$ boolean and.

Here is a more interesting example.

Proposition 4.2.3 *If D is a Scott domain, then*

$$+ : D^2 \to D$$

given by

$$x + y = \begin{cases} x \sqcup y & \text{if } x, y \text{ are consistent;} \\ \bot & \text{otherwise,} \end{cases}$$

is an iterative operation which is idle everywhere. The suspended sum of a consistent sequence is its supremum.

proof First we prove that $+$ is μ continuous. The set of consistent elements

$$C = \{(x, y) \in D^2 : (\,\exists\, z \in D\,)\; x, y \sqsubseteq z\}$$

is a Scott closed subset of the domain D^2. It is easy to see that it is a lower set; that the set is closed under directed suprema requires only the (Lawson) compactness of D.

Indeed, let $S \subseteq C$ be a directed subset. If $(x, y) \in S$, then $\uparrow x \cap \uparrow y$ is a nonempty Scott compact upper set using 1.1.5 and 1.1.4. Then the intersection

$$\bigcap_{(x,y)\in S} (\uparrow x \cap \uparrow y)$$

is filtered by the directedness of S and nonempty by Theorem 1.1.1. Any point z in this intersection satisfies $\bigsqcup S \sqsubseteq (z, z)$ hence $\bigsqcup S \in C$.

Then the definition of $+$ has the form

$$x + y = \begin{cases} f(x, y) & \textit{if} \;\; x, y \in C, \\ g(x, y) & \text{otherwise} \end{cases}$$

where $f : C \to D$ and $g : D^2 \setminus C \to D$ are Scott continuous maps between dcpo's, and C is Scott closed hence μ clopen by Lemma 3.4.2 and thus a subdomain by Theorem 3.4.2. By Corollary 3.4.1, the map $+$ is μ continuous.

To see that $+$ is iterative, let $\sigma(x) = \downarrow x$ for $x \in D$. This set is Scott closed hence open in the μ topology. It is clear that $(\sigma(x), +)$ is a semigroup, so let (x_n) be a sequence in $\sigma(x)$ which converges to x in the μ topology. Then with $s_1 = x_1$ and $s_{n+1} = s_n + x_{n+1}$ for $n \geq 1$, we see that

$$x_n \sqsubseteq s_n \sqsubseteq \bigsqcup_{n \geq 1} x_n = \lim_{n \to \infty} x_n = x$$

for all $n \geq 1$. It now follows easily that

$$\sum_{n \geq 1} x_n = \lim_{n \to \infty} s_n = \bigsqcup_{n \geq 1} x_n = x.$$

This enables the calculation

$$\lim_{n \to \infty} \sum_{k \geq n} x_k = \lim_{n \to \infty} x = x = \lim_{n \to \infty} x_n,$$

which proves that $+$ is idle at every point of D. $\Box$

In the last result, the suspended sum of an inconsistent sequence is $\perp$. If one does not like this, they may map an inconsistent pair (x, y) to y, thereby ensuring that the suspended sum of an inconsistent sequence is the supremum of its largest consistent tail. From here on we shall write the operation of Proposition 4.2.3 as $\sqcup$, instead of $+$. Notice that it is neither monotone nor associative.

Example 4.2.2 Let D be a continuous dcpo.

(i) The operation $+_1 : D^2 \to D$ given by $x +_1 y = x$ is Scott continuous and idle everywhere. The suspended sum of a sequence (x_n) is its first element x_1.

(ii) The operation $+_2 : D^2 \to D$ given by $x +_2 y = y$ is Scott continuous and idle everywhere. The suspended sum of a sequence is the supremum of its largest consistent tail.

In fact, these two may be combined to model 'if' statements.

Proposition 4.2.4 *If V is a μ clopen subset of a domain D, then*

$$+_V : D^2 \to D$$

given by

$$x +_V y = \begin{cases} x & \text{if } x \in V; \\ y & \text{otherwise,} \end{cases}$$

is an iterative operation which is idle everywhere.

proof The only concern here is the μ continuity of $+_V$, whose definition may be written as

$$x +_V y = \begin{cases} f(x,y) & \text{if } (x,y) \in C; \\ g(x,y) & \text{otherwise,} \end{cases}$$

where $C = V \times D$, $f : C \to D$ is $f(x,y) = x$ and $g : D^2 \setminus C \to D$ is $g(x,y) = y$. Now notice that C is a μ clopen subset of the domain D^2 because it is the product of μ clopen subsets of D, and Lemma 4.2.1 ensures that the product of the μ topologies on D is the same as the μ topology on D^2. Then since the maps f and g are obviously Scott continuous as maps between dcpo's, Corollary 3.4.1 gives the μ continuity of $+_V$.

To prove each point is idle, one simply considers the cases $x \in V$ and $x \notin V$ separately. Each is trivial since the sets V and $D \setminus V$ contain sets of the form $\Uparrow a \cap \downarrow x$. $\square$

Example 4.2.3 The inf operation on a Scott domain.

(i) The minimum of two naturals $(\mathbb{N}^{\infty}_{\perp}, \min)$ is idle at all points.

(ii) The inf operation on Σ^{∞} is idle at all points.

(iii) The inf operation on $\mathbf{I}\mathbb{R}$ given by

$$[a, b] \sqcap [c, d] = [\min\{a, c\}, \max\{b, d\}]$$

is idle at all points.

Indeed, for $x \in \mathbf{I}\mathbb{R}$, set $\sigma(x) = \downarrow x$ and consider a sequence (x_n) in $\sigma(x)$ with $x_n \to x$. Write $x = [a, b]$ and $x_n = [a_n, b_n]$. Then the sequence of partial sums is given by

$$s_n = x_1 \sqcap \cdots \sqcap x_n = [\min_{1 \leq i \leq n} a_i, \max_{1 \leq i \leq n} b_i].$$

Now since $\mu x_n \to \mu x$, we know $a_n \to a$ and $b_n \to b$, while the fact that $x_n \in \sigma(x)$ gives $a_n \leq a$ and $b \leq b_n$. Thus, there are integers k_1, k_2 such that

$$a_{k_1} = \min_{n \geq 1} a_n \text{ and } b_{k_2} = \max_{n \geq 1} b_n.$$

This proves that

$$s_n = [a_{k_1}, b_{k_2}]$$

for all $n \geq \max\{k_1, k_2\}$. Then the sequence of partial sums is eventually constant. For the other property, just note

$$\lim_{n \to \infty} \sum_{k \geq n} x_k = \lim_{n \to \infty} [\min_{i \geq n} a_i, \max_{i \geq n} b_i] = [a, b] = \lim_{n \to \infty} x_n.$$

Finally, the operation $\sqcap$ is Scott continuous, so it is μ continuous.

(iv) The intersection operation $(\mathcal{P}\omega, \cap)$ is iterative. However, the only idle points are the finite sets. Let S be an infinite subset of $\mathbb{N}$ whose elements we enumerate as

$$S = \{x_1, x_2, x_3, \cdots\}$$

where $x_n < x_{n+1}$ for all $n \geq 1$. Define $S_n = S \backslash \{x_n\}$ for all $n \geq 1$. First, $S_n \to S$ in the μ topology on $\mathcal{P}\omega$: Using the natural measurement $|\cdot|$ on $\mathcal{P}\omega$ from Example 2.5.1,

$$|S_n| = 1 - \sum_{i \in S_n} \frac{1}{2^{i+1}} = |S| + \frac{1}{2^{x_n+1}} \to |S|,$$

while $S_n \sqsubseteq S$ for all n, so $S_n \to S$ by Proposition 3.1.6. Now observe that the sequence of partial sums associated with *any* tail of (S_n)

$$S_k \cap S_{k+1} \cap \cdots \cap S_{k+n} = S \setminus \{x_i : k \leq i \leq k+n\}$$

has no μ limit because it is a *decreasing* sequence of *distinct* elements.

Notice that $\sqcap$ on $\mathbf{I}\mathbb{R}$ is idle at all points, while $\sqcap$ on $\mathcal{P}\omega$ is only idle at the compact elements. This provides a nice illustration of the nature of iterative operations: An operation is idle at a point x iff all sequences converging to x have infinite sums that may be *reasonably well-approximated* by finite sums. In the case of $\mathcal{P}\omega$, there is no sense in which the intersection $\bigcap_{n\geq 1} S_n$ (a finite set) is well-approximated by its sequence of partial sums (each member of which is infinite).

Example 4.2.4 Operations may fail to idle at points for any number of reasons:

(i) If D has a least element $\perp$, define $x+y=\perp$. The operation is associative, μ continuous and every sequence has an infinite sum. However, it is *not* idle at $x \neq \perp$ because undoing the summation process does not allow us to recover limits:

$$\lim_{n\to\infty} \sum_{k\geq n} x_k = \perp \neq \lim_{n\to\infty} x_n = x.$$

Hence, $+$ is only idle at $\perp$.

(ii) The domain $[0,\infty)^*$ with the usual addition $+$ has no idle points: The only idempotent is 0 and every μ open set around it contains a tail of $\{1/n : n \geq 1\}$. However, even when a sequence does have an infinite sum, it does not converge in the μ topology (the sequence of partial sums is decreasing in the information order).

4.3 Iteration on a Domain

Now we come to what makes iterative operations of interest.

Definition 4.3.1 A splitting $r : D \to D$ on a dcpo D is *inductive* if for all $x \in D$, $\bigsqcup r^n x \in \text{fix}(r)$.

Definition 4.3.2 Let D be a dcpo and $(E,+)$ be a domain with an iterative operation. A function $\delta : D \to E$ *varies* with an inductive map $r : D \to D$ provided that

(i) For all $x \in \text{fix}(r)$, $\delta(x)$ is idle in E, and

(ii) For all $x \in D$, $\delta(r^n x) \to \delta(\bigsqcup r^n x)$ in the μ topology on E.

The function δ interprets the recursive part r of an algorithm in the domain $(E, +)$. A fixed point of r is mapped to an idle point in E: A point where recursion stops.

Definition 4.3.3 Let D be a dcpo and $(E, +)$ be a domain with an iterative operation. A *renee equation* on $D \to E$ is one of the form

$$\varphi = \delta + \varphi \circ r$$

where $r : D \to D$ is inductive and $\delta : D \to E$ varies with r.

Theorem 4.3.1 (Canonicity) *The renee equation*

$$\varphi = \delta + \varphi \circ r$$

has a unique solution which varies with r and agrees with δ on fix(r).

proof Define $\varphi : D \to E$ by the suspended sum

$$\varphi(x) = \overleftarrow{\sum_{n \geq 0}} \delta r^n x.$$

By the associativity in Proposition 4.2.1,

$$\varphi(x) = \overleftarrow{\sum_{n \geq 0}} \delta r^n x = \delta x + \overleftarrow{\sum_{n \geq 1}} \delta r^n x = \delta x + \varphi(rx),$$

and so this is a solution. If x is a fixed point of r, then

$$\varphi(x) = \overleftarrow{\sum_{n \geq 0}} \delta r^n x = \sum_{n \geq 0} \delta x = \delta x,$$

because $\delta(x)$ is idle. This proves that φ agrees with δ on fix(r). To see that it also varies with r, notice that

$$\varphi(r^n x) = \overleftarrow{\sum_{k \geq n}} \delta r^k x,$$

for all $n \geq 0$, and so

$$\lim_{n \to \infty} \varphi(r^n x) = \lim_{n \to \infty} \overleftarrow{\sum_{k \geq n}} \delta r^k x = \lim_{n \to \infty} \delta r^n x = \delta(\bigsqcup r^n x).$$

However, r is inductive, so $\bigsqcup r^n x \in \mathrm{fix}(r)$. Thus, $\delta(\bigsqcup r^n x) = \varphi(\bigsqcup r^n x)$. Now suppose $\lambda = \delta + \lambda \circ r$ varies with r and agrees with δ on $\mathrm{fix}(r)$. Let $x \in D$ and write $r^\infty x := \bigsqcup_{n\geq 0} r^n x$. Because both $(\delta r^n x)_{n\geq 0}$ and $(\lambda r^n x)_{n\geq 0}$ are sequences which converge to $\delta(r^\infty x)$ in the μ topology, there is a least integer $m \geq 0$ such that

$$\delta r^n x, \lambda r^n x \in \sigma(\delta r^\infty x)$$

for all $n \geq m$. Since $\lambda = \delta + \lambda \circ r$, iterating m times yields

$$\lambda x = \delta x + (\delta r x + (\cdots + (\delta r^m x + \lambda r^{m+1} x)\cdots)).$$

Now consider the sequences $(x_i)_{i\geq 0}$ and $(y_i)_{i\geq 0}$, given by

$$\begin{array}{lcl} x_0 & = & \delta r^m x \\ x_{i+1} & = & x_i + \delta r^{m+i+1} x, \quad i \geq 0; \end{array}$$

and $y_i = \lambda r^{m+i+1} x$ for $i \geq 0$. Because each of these are contained in the semigroup $(\sigma(\delta r^\infty x), +)$, we can iterate the equation $m + i$ times to obtain

$$\lambda x = \delta x + (\delta r x + (\cdots + (\delta r^{m-1} x + (x_i + y_i))\cdots)),$$

for all $i \geq 0$. However, (x_i) is the sequence of partial sums for $\sum_{n\geq m} \delta r^n x$, while $\lim y_i = \lambda r^\infty x = \delta r^\infty x$. Then by the μ continuity of $+$, we see that

$$\lambda x = \delta x + (\delta r x + (\cdots + (\delta r^{m-1} x + (\sum_{n\geq m} \delta r^n x + \delta r^\infty x))\cdots)),$$

which in view of Proposition 4.2.2(iii) reduces to

$$\lambda x = \delta x + (\delta r x + (\cdots + (\delta r^{m-1} x + (\sum_{n\geq m} \delta r^n x))\cdots)),$$

which by the choice of m is clearly the suspended sum of $\{\delta r^n x : n \geq 0\}$. □

The importance of φ varying with r is explained by the following result.

Proposition 4.3.1 (Correctness) *Let $\varphi = \delta + \varphi \circ r$ be the unique solution of a renee equation on $D \to E$. If P is any subset of E, then the following are equivalent:*

(i) $(\forall x)\, \varphi(x) \in P$.

(ii) *For every μ open set U containing P,*

  (a) $(\forall x \in \mathrm{fix}(r))\, \delta(x) \in U$, *and*
  (b) $(\forall x)\, \varphi(rx) \in U \Rightarrow \varphi(x) \in U$.

proof (ii) $\Rightarrow$ (i): Let U be any μ open subset of E containing P. If $x \in D$ and $\varphi(x) \in E \setminus U$, then by repeatedly applying (b),

$$(\forall n)\, \varphi(r^n x) \in E \setminus U.$$

The set $E \setminus U$ is μ closed and $\varphi r^n x \to \varphi(\bigsqcup r^n x)$ hence $\varphi(\bigsqcup r^n x) \in E \setminus U$. But $\bigsqcup r^n x$ is a fixed point of the inductive map r and $\varphi = \delta$ on $\mathrm{fix}(r)$, so $\delta(\bigsqcup r^n x) \in E \setminus U$, which contradicts (a). Then $\varphi(x) \in U$ for all $x \in D$. But the μ topology on E is Hausdorff so P is the intersection of all open sets containing it. $\Box$

The last proposition tells us that, in order to prove a solution φ has property P for all inputs, we may verify the induction axioms (a) and (b) for all open sets containing the property P. However, very often we want to prove φ has property P for only a subset I of all possible inputs D. The next proposition is useful in that regard.

Proposition 4.3.2 *Let $\varphi = \delta + \varphi \circ r$ be the unique solution of a renee equation on $D \to E$. If $I \subseteq D$ is a set closed under directed suprema with $r(I) \subseteq I$, then the unique solution of*

$$\lambda = \delta|_I + \lambda \circ r|_I$$

is simply $\varphi|_I$.

proof The restriction of r to I yields an inductive map on I. Similarly, restricting δ to I yields a map which varies with $r|_I$. The solution of the new equation $\lambda = \delta|_I + \lambda \circ r|_I$ must be $\varphi|_I$ by uniqueness. $\Box$

Here are a few basic instances of the renee equation.

Example 4.3.1 The factorial function

$$\text{fac} : \mathbb{N} \to \mathbb{N}$$

is given by

$$\begin{array}{rcl} \text{fac}\, 0 & = & 1 \\ \text{fac}\, n & = & n \times \text{fac}\, n - 1. \end{array}$$

Let $D = \mathbb{N}^*$ and $E = (\mathbb{N}^*, \times)$. Define $\delta : D \to E$ by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ n & \text{otherwise.} \end{cases}$$

and $\text{pred} : D \to D$ by $\text{pred}(n) = n - 1$, if $n > 0$, and $\text{pred}(0) = 0$. The unique solution of

$$\varphi = \delta \times \varphi \circ \text{pred}$$

which satisfies $\varphi(0) = 1$ is the factorial function.

Example 4.3.2 The length of a list

$$\text{len} : [S] \to \mathbb{N}$$

is given by

$$\begin{array}{rcl} \text{len}\, [\,] & = & 0 \\ \text{len}\, a :: x & = & 1 + \text{len}\, x. \end{array}$$

Let $D = [S]$ and $E = (\mathbb{N}^*, +)$. Define $\delta : D \to E$ by

$$\delta(x) = \begin{cases} 0 & \text{if } x = [\,], \\ 1 & \text{otherwise.} \end{cases}$$

and $\text{rest} : D \to D$ by $\text{rest}(a :: x) = x$ and $\text{rest}([\,]) = [\,]$. The unique solution of

$$\varphi = \delta + \varphi \circ \text{rest}$$

which satisfies $\varphi([\,]) = 0$ is the length function.

The last example shows that there are measurements which may be realized as the unique solutions of renee equations.

Example 4.3.3 The linear search of a list x for a key k

$$\text{search} : [S] \times S \to \{\bot, \top\}$$

is given by

$$\begin{array}{lcll} \text{search}([\,], k) & = & \bot & \\ \text{search}(x, k) & = & \top & \text{if first } x = k, \\ \text{search}(x, k) & = & \text{search}(\text{rest } x, k) & \text{otherwise.} \end{array}$$

Let $D = [S] \times S^\flat$ and $E = (\{\bot, \top\}^\flat, \vee)$. Define $\delta : D \to E$ by

$$\delta(x, k) = \begin{cases} \bot & \text{if } x = [\,], \\ \top & \text{if first } x = k, \\ \bot & \text{otherwise.} \end{cases}$$

and $r : D \to D$ by $r(x, k) = (\text{rest } x, k)$. The unique solution of

$$\varphi = \delta \vee \varphi \circ r$$

which satisfies $\varphi([\,], k) = \bot$ for all k is search.

Example 4.3.4 The merging of two sorted lists of integers

$$\text{merge} : [\text{int}] \times [\text{int}] \to [\text{int}]$$

is given by the following ML code

```
fun merge( [ ], ys )          = ys : int list
  | merge( xs, [ ] )          = xs
  | merge( x :: xs, y :: ys ) = if x <= y then
                                   x :: merge( xs, y :: ys )
                                else
                                   y :: merge( x :: xs, ys );
```

Let $D = [\text{int}] \times [\text{int}]$ and $E = ([\text{int}], \cdot)$. Define $\delta : D \to E$ by

$$\begin{array}{lcll} \delta(x, [\,]) & = & x & \\ \delta([\,], y) & = & y & \\ \delta(x, y) & = & [\min(\text{first } x, \text{first } y)], & \text{otherwise.} \end{array}$$

and $\pi : D \to D$ by

$$\begin{array}{lcll}
\pi(x, [\,]) & = & ([\,], [\,]) & \\
\pi([\,], y) & = & ([\,], [\,]) & \\
\pi(x, y) & = & (\text{rest}\, x, y), & \text{if first}\, x \leq \text{first}\, y; \\
\pi(x, y) & = & (x, \text{rest}\, y), & \text{otherwise.}
\end{array}$$

The unique solution of

$$\varphi = \delta \cdot \varphi \circ \pi$$

satisfying $\varphi([\,], [\,]) = [\,]$ is merge.

The last example is interesting because solving the equation yields a new iterative operation on [int]. We shall make use of this fact in the next example to solve an equation for sorting. In this way, we see that algorithms can be built up by solving sequences of renee equations.

Example 4.3.5 The prototypical bubblesort of a list of integers

$$\text{sort} : [\text{int}] \to [\text{int}]$$

is given by

$$\begin{array}{lcl}
\text{sort}\, [\,] & = & [\,] \\
\text{sort}\, x & = & \text{merge}(\, [\text{first}\, x],\ \text{sort rest}\, x\ )
\end{array}$$

Let $D = [\text{int}]$ and $E = ([\text{int}], +)$ where

$$+ : [\text{int}]^2 \to [\text{int}]$$

$$(x, y) \mapsto \text{merge}(x, y)$$

is the merge operation of Example 4.3.4. Define $\delta : D \to E$ by

$$\delta(x) = \begin{cases} [\,] & \text{if } x = [\,] \\ [\text{first}\, x] & \text{otherwise} \end{cases}$$

and let rest : [int] $\to$ [int] be the usual splitting. The unique solution of

$$\varphi = \delta + \varphi \circ \text{rest}$$

satisfying $\varphi[\,] = [\,]$ is sort.

We now illustrate the correctness technique implied by Propositions 4.3.1 and 4.3.2. Notice that all of the domains in the previous examples have discrete μ topology, so the same ideas are easily applied there as well.

Example 4.3.6 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map on the real line $\mathbb{R}$ and $C(f) := \{ [a, b] \in \mathbf{I}\mathbb{R} : f(a) \cdot f(b) \leq 0 \}$. The bisection algorithm,

$$\text{bisect} : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R},$$

for $\varepsilon > 0$ fixed, is given by

$$\begin{array}{lcll} \text{bisect}\,[a,b] & = & [a,b] & \text{if } \mu[a,b] < \varepsilon; \\ \text{bisect}\,[a,b] & = & \text{bisect left}[a,b] & \text{if left}[a,b] \in C(f); \\ \text{bisect}\,[a,b] & = & \text{bisect right}[a,b] & \text{otherwise.} \end{array}$$

Define an iterative operation $+_\varepsilon$ on $\mathbf{I}\mathbb{R}$ by

$$x +_\varepsilon y = \begin{cases} x & \text{if } \mu x < \varepsilon; \\ y & \text{otherwise,} \end{cases}$$

using Proposition 4.2.4 and the standard measurement $\mu : \mathbf{I}\mathbb{R} \to [0, \infty)^*$. Let $D = \mathbf{I}\mathbb{R}$ and $E = (\mathbf{I}\mathbb{R}, +_\varepsilon)$. Finally, define $\delta : D \to E$ to be the identity map and $\text{split}_f : D \to D$ by

$$\text{split}_f[a,b] = \begin{cases} \text{left}[a,b] & \text{if left}[a,b] \in C(f); \\ \text{right}[a,b] & \text{otherwise.} \end{cases}$$

The unique solution of

$$\varphi = \delta +_\varepsilon \varphi \circ \text{split}_f$$

which satisfies $\varphi[r] = [r]$ for every $f(r) = 0$ is bisect. What we want to prove about φ is that

$$(\forall x \in C(f))\, \varphi(x) \in C(f) \;\&\; \mu\varphi(x) < \varepsilon.$$

Then we should let $I = C(f)$ and $P = C(f) \cap \mu^{-1}[0, \varepsilon)$. Since $\text{split}_f(I) \subseteq I$, we can apply Proposition 4.3.1 to $\varphi|_I$. If $U \subseteq \mathbf{I}\mathbb{R}$ is a μ open set containing the property P, then

(i) $(\forall x \in I \cap \text{fix}(\text{split}_f))\, \varphi(x) = \delta(x) = x \in P \subseteq U$, and

(ii) $(\forall x \in I)\, \varphi(\text{split}_f x) \in U \Rightarrow \varphi(x) \in U$,

where we note that (i) holds because the fixed points of split_f all have measure zero, while (ii) holds using the fact that

$$\varphi(x) = \begin{cases} x & \text{if } \mu x < \varepsilon; \\ \varphi(\text{split}_f x) & \text{otherwise.} \end{cases}$$

By Proposition 4.3.1, $\varphi|_I(x) \in P$, for all $x \in I$.

In the last example, we chose to have the operation $+_\varepsilon$ provide the test for termination. Another possibility is to use the $\sqcup$ operation and modify split_f so that it maps intervals of length less than ε to themselves.

Example 4.3.7 The Riemann integral. Let $f : [a, b] \to \mathbb{R}$ be a continuous map and let $\mathcal{P}[a, b] = \{P_n : n \geq 1\} \cup \{[a, b]\}$ be the partitions of $[a, b]$ into n equal parts, ordered as an increasing chain

$$P_1 \sqsubseteq P_n \sqsubseteq P_{n+1} \sqsubseteq [a, b]$$

for all n. The correspondence

$$P_n \mapsto [L_f(P_n), U_f(P_n)]$$

defines a monotone mapping from the compact elements of the algebraic domain $\mathcal{P}[a, b]$ into the domain $\mathbf{I}\mathbb{R}$. Hence, it has a unique Scott continuous extension

$$\delta : \mathcal{P}[a, b] \to \mathbf{I}\mathbb{R}$$

The map $r : \mathcal{P}[a, b] \to \mathcal{P}[a, b]$ given by $r(P_n) = P_{n+1}$, and $r([a, b]) = [a, b]$ is inductive, so the equation

$$\varphi = \delta \sqcup \varphi \circ r$$

has a unique solution satisfying $\varphi[a, b] = \delta[a, b]$. Finally, as an example, notice that the value of φ at the finite element $x = \{a, b\} \in \mathcal{P}[a, b]$ is computed as

$$\varphi(x) = \overleftarrow{\sum_{n\geq 0}} \delta r^n x = \bigsqcup_{n\geq 0} \delta r^n x = \bigcap_{n\geq 1} [L_f(P_n), U_f(P_n)] = \left[\int_a^b f(x)\, dx\right]$$

From this we see that sometimes renee equations may define *objects*.

Example 4.3.8 Let $f : D \to D$ be a Scott continuous map on a domain with $I(f) = \{x \in D : x \sqsubseteq f(x)\} \neq \emptyset$. Define $r : I(f) \to I(f)$ by $r(x) = f(x)$, and $\delta : I(f) \to D$ by $\delta(x) = f(x)$. The unique solution of

$$\varphi = \delta +_2 \varphi \circ r = \varphi \circ r$$

which satisfies $\varphi(x) = x$ for all $x \in \text{fix}(f)$ is

$$\varphi(x) = \bigsqcup_{n\geq 0} f^n(x).$$

4.4 Algorithm versus Function

The philosophy behind the renee equation is that an algorithm is defined by three things: A *simple* part δ, a *recursive* part r, and an operation $+$ which adds information. The way an algorithm computes its output, given input x, is by first reducing the problem to a stage where it is easily solved (generally this is also when a suspended sum becomes associative). This stage is reached in finite time and is represented by an iterate $r^n x$. At this point, δ is applied to $r^n x$ and then $r^{n-1}x$ and so on to produce a sequence of partial outputs $(\delta r^n x)$. Finally the partial outputs are all added together to give the output $\overleftarrow{\sum}_{n\geq 0} \delta r^n x$. As for the terms from $n+1$ onward, we know in reality that they are essentially negligible, either because we reach an idle point in finite time (as with list algorithms), or because we simply cut off the algorithm for the purposes of approximation (the bisection).

There are a few points worth emphasizing here. First, we are *not* saying that every renee equation defines an algorithm. What seems plausible, though, is that every nontrivial algorithm (one that makes use of iteration) is the unique solution of a renee equation. One should not, however, attempt to separate the solution from the equation which defines it. To speak reasonably about such things, one must say that an algorithm is the function *φ whose output is calculated using the equation $\varphi = \delta + \varphi \circ r$.* Here is one possible definition of what this means.

Definition 4.4.1 For a splitting $r : D \to D$, define

$$\lambda_r : D \to \mathbb{N}^\infty$$

$$\lambda_r x = \inf\{n \in \mathbb{N} : r^n x = r^{n+1} x\}$$

with the convention that $\lambda_r x = \infty$ if the set above is empty.

Thus, $\lambda_r x$ is the number of times that the splitting r loops on input x.

Definition 4.4.2 The map computed by $\varphi = \delta + \varphi \circ r$ on $D \to E$ is

$$\bar{\varphi} : D \to E$$

$$\bar{\varphi} x = \begin{cases} \delta x + (\delta r x + (\cdots + (\delta r^{\lambda_r x} x) \cdots)) & \text{if } \lambda_r x < \infty; \\ \infty & \text{otherwise;} \end{cases}$$

where we assume E equipped with a symbol ∞ for nontermination.

A safe intuition here is that φ is computable if $\delta, +$, and r are computable and $\varphi = \bar{\varphi}$. The next example shows one must be careful in trying to say much else.

Example 4.4.1 Consider the equation

$$\varphi(x) = \min\{x, \varphi(x+1)\}$$

on $\mathbb{N}^{\infty} \to \mathbb{N}^{\infty}$. The solution is $\varphi = 1_{\mathbb{N}^{\infty}}$, while $\bar{\varphi} = \infty$. Hence, $\bar{\varphi}$ is the function we'd get were we to code the equation on a machine, while φ is a mathematical object guaranteed to exist uniquely.

Even though both functions in the last example are computable, an important issue arises: We would not only like to capture algorithms as unique solutions of renee equations, we would also like these solutions to be calculable *using the equation,* that is, we would like every algorithm to satisfy $\varphi = \delta + \varphi \circ r$ with $\varphi = \bar{\varphi}$.

Notice that such a description is programming language independent. We have only made use of abstract properties all domains possess: Iterative operations, inductive maps, the μ topology. If there is any reference to a language at all, it is to the intrinsic language that every domain may possess, one whose grammar might include productions like

$$p ::= \delta \mid \varphi = p + \varphi \circ r.$$

Finally, note that the form of the renee equation is very specific: The recursive part of an algorithm takes place only in D (the space of inputs), while information can only be added together in E (the space of outputs). It may seem that such a form could not seriously describe very much.

4.5 The Computable Functions on an Arithmetic Domain

Let D be a domain which as a set satisfies $\mathbb{N} \subseteq D \subseteq \mathbb{N} \cup \{\infty\}$. Three examples of such a domain are $\mathbb{N}^{\infty}, \mathbb{N}^{*}$, and $\mathbb{N}^{\flat}$.

Definition 4.5.1 The sequence of domains $(D^n)_{n \geq 1}$ is given inductively by

$$\begin{aligned} D^1 &= D, \\ D^{n+1} &= D^n \times D^1, \quad n > 0. \end{aligned}$$

Mappings of type $D^n \to D$ can in general be thought of as partial mappings on $\mathbb{N}$.

Definition 4.5.2 Given a mapping $f : D^n \to D$, its *normalization*

$$|f| : \mathbb{N}^n \to \mathbb{N}_\perp$$

is defined by

$$|f|(x) = \begin{cases} f(x) & \text{if } f(x) \in \mathbb{N}; \\ \perp & \text{otherwise.} \end{cases}$$

The normalization of a map $r : D^n \to D^n$ is then

$$|r| : \mathbb{N}^n \to (\mathbb{N}_\perp)^n$$

given by

$$|r|(x) = (|r_1|(x), \cdots, |r_n|(x))$$

where we write $r = (r_1, \cdots, r_n)$ in terms of its *coordinates* $r_i : D^n \to D$, for $1 \le i \le n$.

Then to discuss computability on D it helps to keep in mind that all maps $D^n \to D$ represent partial maps $\mathbb{N}^n \to \mathbb{N}_\perp$. By Kleene's Theorem 4.1.1, we see that analogues of initial function, composition and recursion on $\mathbb{N}$ must be developed for D. First we choose some convenient initial functions.

Definition 4.5.3 The initial functions.

(i) Addition of naturals $+ : D^2 \to D$ given by

$$(x, y) \mapsto \begin{cases} x + y & \text{if } x, y \in \mathbb{N}; \\ \infty & \text{otherwise.} \end{cases}$$

(ii) Subtraction of naturals $\ominus : D^2 \to D$ given by

$$(x, y) \mapsto \begin{cases} \max\{x - y, 0\} & \text{if } x, y \in \mathbb{N}; \\ \infty & \text{otherwise.} \end{cases}$$

(iii) Multiplication of naturals $\times : D^2 \to D$ given by

$$(x, y) \mapsto \begin{cases} x \times y & \text{if } x, y \in \mathbb{N}; \\ \infty & \text{otherwise.} \end{cases}$$

(iv) The predicate $\leq: D^2 \to D$ given by

$$(x, y) \mapsto \begin{cases} x \leq y & \text{if } x, y \in \mathbb{N}; \\ \infty & \text{otherwise.} \end{cases}$$

(v) The projections $\pi_i^n : D^n \to D$, for $n \geq 1$ and $1 \leq i \leq n$, given by

$$(x_1, \cdots, x_n) \mapsto \begin{cases} x_i & \text{if } (x_1, \cdots, x_n) \in \mathbb{N}^n; \\ \infty & \text{otherwise.} \end{cases}$$

It is just as easy to see that there will be no difficulty in representing composition of mappings on D. However, how do we handle recursion? For this, we simply take closure under renee equations.

Definition 4.5.4 Let $\mathcal{C}(D)$ be the smallest class of functions $f : D^n \to D$ with the following properties:

(i) $\mathcal{C}(D)$ contains $+$, $\ominus$, $\times$, $\leq$, and π_i^n, for $n \geq 1$ and $1 \leq i \leq n$,

(ii) $\mathcal{C}(D)$ is closed under substitution: If $f : D^n \to D$ is in $\mathcal{C}(D)$ and $g_i : D^k \to D$ is in $\mathcal{C}(D)$, for $1 \leq i \leq n$, then

$$f(g_1, \cdots, g_n) : D^k \to D \in \mathcal{C}(D),$$

and

(iii) $\mathcal{C}(D)$ is closed under iteration: If $\delta : D^n \to D$ and $+ : D^2 \to D$ are in $\mathcal{C}(D)$, and $r : D^n \to D^n$ is a map whose coordinates are in $\mathcal{C}(D)$, then

$$\varphi = \delta + \varphi \circ r \in \mathcal{C}(D)$$

whenever this is a renee equation on $D^n \to D$.

In the next three sections we substantiate the claim that what $\mathcal{C}(D)$ describes is the set of computable functions on the arithmetic domain D.

4.6 The Partial Recursive Functions

In this section we set $D = \mathbb{N}^\infty$, $\mathcal{C} = \mathcal{C}(D)$, and prove that $|\mathcal{C}|$ is the class of partial recursive functions on the naturals.

Definition 4.6.1 An element $x \in D^n \setminus \mathbb{N}^n$ is called *undefined.* The unique maximal element of D^n is written ∞^n.

Definition 4.6.2 A *partial mapping* on D is a function $f : D^n \to D$ which preserves undefined elements.

This definition is natural as follows.

Definition 4.6.3 A partial mapping $f : \mathbb{N}^n \to \mathbb{N}_\perp$ extends naturally to a map $\bar{f} : D^n \to D$ via

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{N}^n \ \& \ f(x) \in \mathbb{N}; \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\bar{f}$ preserves undefined elements.

Proposition 4.6.1 *The following properties hold:*

(i) *For a partial map* $f : \mathbb{N}^n \to \mathbb{N}_\perp$, $|\bar{f}| = f$.

(ii) *For a partial map* $f : D^n \to D$, $\overline{|f|} = f$.

(iii) *The correspondence* $f \mapsto \bar{f}$ *is a bijection between partial mappings on* $\mathbb{N}$ *and partial mappings on* D.

Thus, we have trivially represented each partial function $f : \mathbb{N}^n \to \mathbb{N}_\perp$ as a mapping $\bar{f} : D^n \to D$ between domains. The question we entertain is as follows: What domain theoretic construction exactly captures the notion of partial recursive function?

Lemma 4.6.1 *The following basic properties hold:*

(i) *Every* $f \in \mathcal{C}$ *is a partial mapping.*

(ii) *If* $r : D^n \to D^n$ *is a function whose coordinates are in* $\mathcal{C}$, *then*

$$\mathrm{fix}(r) \subseteq \mathbb{N}^n \cup \{\infty^n\}.$$

proof (i) Every function $f : D^n \to D$ in $\mathcal{C}$ has the property that

$$x \in D^n \setminus \mathbb{N}^n \Rightarrow f(x) = \infty,$$

as one easily sees by a structural induction on the class $\mathcal{C}$. (ii) Write r in terms of its coordinates $r = (r_1, \cdots, r_n)$, where $r_i : D^n \to D$, for $1 \leq i \leq n$. If r has an undefined fixed point x,

$$x = r(x) = (r_1(x), \cdots, r_n(x)) = (\infty, \cdots, \infty) = \infty^n,$$

where we have applied (i) to the r_i. $\square$

Lemma 4.6.2 *If $f : D^n \to D$ is in $\mathcal{C}$ and $g_i : D^k \to D$, for $1 \leq i \leq n$, are in $\mathcal{C}$, then*

$$|f(g_1, \cdots, g_n)| = |f|(|g_1|, \cdots, |g_n|).$$

proof This is immediate from Lemma 4.6.1(i). Notice, though, that the expression on the right is composition of partial mappings. $\square$

Notice that we do not need to assume $\ominus$ is an initial function when $D = \mathbb{N}^\infty$.

Lemma 4.6.3 *The following functions belong to $\mathcal{C}$:*

(i) *The constant functions* 0 *and* 1.

(ii) *The succesor* $\mathrm{succ} : D \to D$ *given by* $\mathrm{succ}(x) = x + 1$.

(iii) *The predicates* $<$, $=$, *and* $>$.

(iv) *The subtraction function* $\ominus : D^2 \to D$ *given by*

$$x \ominus y = \begin{cases} x - y & \text{if } x > y \ \& \ x, y \in \mathbb{N}; \\ 0 & \text{if } x \leq y \ \& \ x, y \in \mathbb{N}; \\ \infty & \text{otherwise.} \end{cases}$$

proof Equality given by $a = b := (a \leq b) \times (b \leq a)$ is in $\mathcal{C}$ by substitution. Thus, $\mathrm{one}(x) := (x = x)$ belongs to $\mathcal{C}$. Then $\mathrm{succ}(x) = x + \mathrm{one}(x)$ also belongs to $\mathcal{C}$, which proves (ii). Now we can finish the proof of (i) since

$$\mathrm{zero}(x) := (x = \mathrm{succ}(x)) = \begin{cases} 0 & \text{if } x \in \mathbb{N}; \\ \infty & \text{otherwise.} \end{cases}$$

must belong to $\mathcal{C}$. For (iii), $\mathrm{not}(x) := (x = 0)$ is in $\mathcal{C}$, so

$$a < b := (a \leq b) \times \mathrm{not}(a = b)$$

is in $\mathcal{C}$ hence $a > b := (b < a)$ is in $\mathcal{C}$. For (iv), we solve a renee equation as follows. Define $\delta : D^3 \to D$ by $\delta(x, y, i) = i + y < x$ and a splitting $r : D^3 \to D^3$ by

$$r(x, y, i) = (x, y, i + (i + y < x)).$$

Notice that r iterates to a fixed point in finite time for any input, and that its coordinates are all in $\mathcal{C}$. As our iterative operation, we take addition of naturals $+ : D^2 \to D$: A Scott hence μ continuous operation whose only idle points are 0 and ∞. Finally, to show δ varies r, we must show fixed points map to idle points under δ since $\delta(r^n x) \to \delta(\bigsqcup r^n x)$ is clear. But

$$r(\bar{x}) = \bar{x} \Rightarrow (\ \bar{x} = (x, y, i)\ \&\ (i + y < x) = 0\ ) \text{ or } \bar{x} = \infty^3,$$

and in these cases, we have $\delta(x, y, i) = 0$ and $\delta(\infty^3) = \infty$, respectively. Then we have a renee equation $\varphi = \delta + \varphi \circ r \in \mathcal{C}$, and so $\varphi(x, y, 0) \in \mathcal{C}$ by substitution. But $x \ominus y = \varphi(x, y, 0)$, which finishes the proof. $\square$

Proposition 4.6.2 *Every partial recursive function is in $|\mathcal{C}|$.*

proof The class $|\mathcal{C}|$ contains the initial functions $+$,$\times$, π^n_i, and $\leq$, while Lemma 4.6.2 shows that $|\mathcal{C}|$ is closed under composition of partial functions.

Now we show that $|\mathcal{C}|$ is closed under minimization. If $f : \mathbb{N}^{n+1} \to \mathbb{N}_\perp$ is in $|\mathcal{C}|$, then $\bar{f} \in \mathcal{C}$, according to Proposition 4.6.1. Let $\delta : D^{n+2} \to D$ be projection on to the $(n + 1)^{st}$ coordinate

$$\delta(\bar{x}, y, z) = y.$$

Define the components of a splitting $r : D^{n+2} \to D^{n+2}$ by

$$r_i : D^{n+2} \to D$$

$$r_i(\bar{x}, y, z) = \pi^{n+2}_i(\bar{x}, y, z) + (\text{eval}_f(\bar{x}, y, z) > 0),$$

for $1 \leq i \leq n + 2$, where $\text{eval}_f : D^{n+2} \to D$ is

$$\text{eval}_f(x_1, \cdots, x_n, y, z) = \bar{f}(x_1 \ominus z, \cdots, x_n \ominus z, y)$$

using the subtraction operation in Lemma 4.6.3(iv). Now, given a vector $\bar{x} = (x_1, \cdots, x_n)$, write $\bar{x} + 1$ for the vector $(x_1 + 1, \cdots, x_n + 1)$, with similar notation for $\bar{x} \ominus z$. Then $r = (r_1, \cdots, r_n) : D^{n+2} \to D^{n+2}$ is given by

$$r(\bar{x}, y, z) = \begin{cases} (\bar{x}, y, z) & \text{if } \bar{f}(\bar{x} \ominus z, y) = 0; \\ (\bar{x} + 1, y + 1, z + 1) & \text{if } \infty \neq \bar{f}(\bar{x} \ominus z, y) > 0; \\ \infty^{n+2} & \text{otherwise.} \end{cases}$$

Notice that the coordinates of r belong to $\mathcal{C}$ and that it iterates to a fixed point for any input. As our iterative operation, we take the sup operation $\max : D^2 \to D$ given by

$$\max\{a, b\} = (a \leq b) \times b + a \times (b < a).$$

The operation max is idle at all points and δ varies with r, so we have a renee equation

$$\varphi = \max\{\delta, \varphi \circ r\} \in \mathcal{C}.$$

Then $\mu f(\bar{x}) = \varphi(\bar{x}, 0, 0)$ is in $\mathcal{C}$ by substitution. Hence, the minimization of f, $|\mu f| : \mathbb{N}^n \to \mathbb{N}_\perp$, is in $|\mathcal{C}|$. Since $|\mathcal{C}|$ contains the initial functions and is closed under composition and minimization, it must contain the partial recursive functions by Theorem 4.1.1. □

Now for the converse.

Lemma 4.6.4 *If $\delta : \mathbb{N}^n \to \mathbb{N}_\perp$ is partial recursive, and $r : \mathbb{N}^n \to (\mathbb{N}_\perp)^n$ is a function whose coordinates are partial recursive, then*

$$f : \mathbb{N}^{n+1} \to \mathbb{N}_\perp$$

$$f(\bar{x}, k) = \delta(r^k(\bar{x}))$$

is partial recursive.

proof Let $b_1 : \mathbb{N} \to \mathbb{N}$ be the identity map and let $b_2 : \mathbb{N}^2 \to \mathbb{N}$ denote any total recursive isomorphism whose inverse is also recursive (which means that $\pi_1^2 \circ b_2^{-1}, \pi_2^2 \circ b_2^{-1} : \mathbb{N} \to \mathbb{N}$ are recursive). For example, one can take

$$b_2(x, y) = 2^x(2y + 1) - 1.$$

For $n \geq 2$, we define a recursive isomorphism

$$b_{n+1} : \mathbb{N}^{n+1} \to \mathbb{N}$$

$$b_{n+1}(\bar{x}, y) = b_2(b_n(\bar{x}), y),$$

with recursive inverse $b_{n+1}^{-1} : \mathbb{N} \to \mathbb{N}^{n+1}$ given by

$$b_{n+1}^{-1}(s) = (b_n^{-1} \circ \pi_1^2 \circ b_2^{-1}(s), \pi_2^2 \circ b_2^{-1}(s)).$$

Now define $s : \mathbb{N} \to \mathbb{N}_\perp$ by $s(x) = b_n \circ r \circ b_n^{-1}(x)$. This function is partial recursive by composition. Furthermore, notice that $s^k(x) = b_n \circ r^k \circ b_n^{-1}(x)$. But calculating $s^k(x)$ is easy:

$$i : \mathbb{N}^2 \to \mathbb{N}_\perp$$

$$i(x,k) = \begin{cases} x & \text{if } k = 0; \\ s(i(x,k-1)) & \text{if } k > 0. \end{cases}$$

The function i arises by applying a primitive recursive scheme to the partial recursive s. Hence, i is partial recursive. Finally, we see that $f : \mathbb{N}^{n+1} \to \mathbb{N}_\perp$ is expressible as

$$f(\bar{x}, k) = \delta(b_n^{-1}(i(b_n(\bar{x}), k))),$$

which proves that it is partial recursive. $\square$

Lemma 4.6.5 *If $\varphi = \delta + \varphi \circ r$ is a renee equation with coefficients in $\mathcal{C}$, then*

$$\lambda_r x = \infty \Rightarrow \varphi(x) = \infty,$$

for all x.

proof Let $\delta : D^n \to D$ and $+ : D^2 \to D$ belong to $\mathcal{C}$, and suppose the coordinates of $r : D^n \to D^n$ are also in $\mathcal{C}$. Now write the k-th iterate as

$$r^k x = (a_k^1, \cdots, a_k^n)$$

for n different increasing sequences $(a_k^i)_{k \geq 0}$, with $1 \leq i \leq n$. If $\lambda_r x = \infty$, one of these sequences is strictly increasing. Without loss of generality, suppose it is $(a_k^1)_{k \geq 0}$. Then

$$\bigsqcup_{k \geq 0} r^k x = (\infty, \bigsqcup_{k \geq 0} a_k^2, \cdots, \bigsqcup_{k \geq 0} a_k^n) \in \text{fix}(r).$$

By Lemma 4.6.1(ii), we must have $\bigsqcup r^k(x) = \infty^n$. Next recall that φ is given explicitly by a suspended sum

$$\varphi(x) = \overleftarrow{\sum_{n \geq 0}} \delta r^n x,$$

as in the proof of Theorem 4.3.1. Hence, there is an integer $n \geq 0$ such that

$$\varphi(r^n x) = \sum_{k \geq n} \delta r^k x,$$

but according to Proposition 4.2.2(iii), we may calculate as follows

$$\begin{aligned}\varphi(r^n x) &= \sum_{k\geq n} \delta r^k x \\ &= (\sum_{k\geq n} \delta r^k x) + \lim_{k\to\infty} \delta r^k x \\ &= (\sum_{k\geq n} \delta r^k x) + \delta(\infty^n) \\ &= (\sum_{k\geq n} \delta r^k x) + \infty \;=\; \infty,\end{aligned}$$

where we have made use of the fact that δ varies with r and the fact that mappings in $\mathcal{C}$ preserve undefined elements. But now we iterate to obtain

$$\varphi(x) = \delta x + (\delta r x + (\cdots + (\delta r^{n-1} x + \varphi(r^n x))\cdots)) = \infty.$$

□

Proposition 4.6.3 *Every function in $|\mathcal{C}|$ is partial recursive.*

proof Let $\mathcal{A} = \{f \in \mathcal{C} : |f| \text{ is partial recursive}\}$. We will prove that $\mathcal{A}$ contains the initial functions and is closed under substitution and iteration. It then follows that we must have $\mathcal{C} \subseteq \mathcal{A}$ since $\mathcal{C}$ is the smallest such class.

$\mathcal{A}$ contains $+$, $\ominus$, $\times$, $\leq$ and π_i^n, for $n \geq 1$ and $1 \leq i \leq n$, since each of the former has a partial recursive normalization.

$\mathcal{A}$ is closed under substitution: If $f : D^n \to D$ is in $\mathcal{A}$ and $g_i : D^k \to D$ is in $\mathcal{A}$, for $1 \leq i \leq n$, then $f(g_1, \cdots, g_n) \in \mathcal{C}$, and by Lemma 4.6.2,

$$|f(g_1, \cdots, g_n)| = |f|(|g_1|, \cdots, |g_n|),$$

but the expression on the right is partial recursive since $f, g_i \in \mathcal{A}$. Hence, $f(g_1, \cdots, g_n) \in \mathcal{A}$.

Finally, $\mathcal{A}$ is closed under iteration. Let $\varphi = \delta + \varphi \circ r$ be a renee equation, where $\delta : D^n \to D$ and $+ : D^2 \to D$ are in $\mathcal{A}$, and $r : D^n \to D^n$ is a map whose coordinates are in $\mathcal{A}$. To show that $\varphi \in \mathcal{A}$ we must show that $|\varphi|$ is partial recursive.

First, $|r| : \mathbb{N}^n \to (\mathbb{N}_\perp)^n$ has partial recursive coordinates, so Lemma 4.6.4 ensures that

$$p_i : \mathbb{N}^{n+1} \to \mathbb{N}_\perp$$
$$p_i(\bar{x}, k) = \pi_i^n(|r|^k(\bar{x}))$$

defines a partial recursive map, for each $1 \leq i \leq n$. Next, the map

$$\Lambda_r : \mathbb{N}^n \to \mathbb{N}_\perp$$

$$\Lambda_r(\bar{x}) = \min\{k \in \mathbb{N} : \bigwedge_{i=1}^{n} p_i(\bar{x}, k) = p_i(\bar{x}, k+1)\},$$

is partial recursive because it is defined by minimization. By Lemma 4.6.4,

$$f : \mathbb{N}^{n+1} \to \mathbb{N}_\perp$$

$$f(\bar{x}, k) = |\delta|(|r|^k(\bar{x}))$$

is partial recursive. Finally,

$$\sigma : \mathbb{N}^{n+1} \to \mathbb{N}_\perp$$

$$\sigma(\bar{x}, k) = \begin{cases} f(\bar{x}, \Lambda_r(\bar{x})) & \text{if } k = 0; \\ |+|(\ f(\bar{x}, \Lambda_r(\bar{x}) \ominus k), \sigma(\bar{x}, k-1)\ ) & \text{if } k > 0; \end{cases}$$

is partial recursive, as it is defined using the primitive recursion scheme. Notice that the operation $x \ominus y = 0$, if $y \geq x$, and $x \ominus y = x - y$, otherwise, is partial recursive. Now we prove that

$$|\varphi|(\bar{x}) = \sigma(\bar{x}, \Lambda_r(\bar{x}))$$

for all $\bar{x} \in \mathbb{N}^n$, by considering the cases $\Lambda_r(\bar{x}) = \perp$ and $\Lambda_r(\bar{x}) \neq \perp$ separately.

Suppose that $\Lambda_r(\bar{x}) = \lambda \in \mathbb{N}$ is defined. Then

$$\begin{aligned} (\forall i)\ p_i(\bar{x}, \lambda) = p_i(\bar{x}, \lambda+1) \neq \perp \quad & \Rightarrow \quad (\forall i)\ \pi_i^n(|r|^\lambda(\bar{x})) = \pi_i^n(|r|^{(\lambda+1)}(\bar{x})) \\ & \Rightarrow \quad |r|^\lambda(\bar{x}) = |r|^{(\lambda+1)}(\bar{x}) \\ & \Rightarrow \quad r^\lambda(\bar{x}) = r^{(\lambda+1)}(\bar{x}) \in \text{fix}(r) \cap \mathbb{N}^n \end{aligned}$$

Thus, $\Lambda_r(\bar{x}) = \lambda \in \mathbb{N} \Leftrightarrow \lambda$ is the least integer with $r^\lambda(\bar{x}) = r^{(\lambda+1)}(\bar{x}) \in \mathbb{N}$. Notice that this implies $r^i(\bar{x}) \in \mathbb{N}$, for all $i \leq \lambda$. Then for a given $\bar{x} \in \mathbb{N}^n$ with $\Lambda_r(\bar{x}) \neq \perp$, we have the following:

(i) $\Lambda_r(\bar{x}) > 0 \Rightarrow \Lambda_r(r\bar{x}) = \Lambda_r(\bar{x}) - 1$,

(ii) $(\forall k \in \mathbb{N})\ k < \Lambda_r(\bar{x}) \Rightarrow f(\bar{x}, \Lambda_r(\bar{x}) - k) = f(r\bar{x}, \Lambda_r(r\bar{x}) - k)$,

(iii) $(\forall k \in \mathbb{N})\ k < \Lambda_r(\bar{x}) \Rightarrow \sigma(\bar{x}, k) = \sigma(r\bar{x}, k)$, and

(iv) $\Lambda_r(\bar{x}) > 0 \Rightarrow \sigma(x, \Lambda_r(\bar{x})) = |+|(\ |\delta|(\bar{x}), \sigma(r\bar{x}, \Lambda_r(r\bar{x}))\ )$

One proves (iii) by induction using (ii) and (i); (iv) then follows from (iii). Now consider the predicate

$$P(k) = (\forall \bar{x} \in \mathbb{N}^n)\, \Lambda_r(\bar{x}) = k \Rightarrow |\varphi|(\bar{x}) = \sigma(\bar{x}, \Lambda_r(\bar{x})).$$

We prove by induction that $(\forall k \in \mathbb{N})\, P(k)$. If $k = 0$ and $\Lambda_r(\bar{x}) = k$, then $r\bar{x} = \bar{x}$ is a fixed point of r, so $\varphi(\bar{x}) = \delta(\bar{x})$. But notice that

$$\sigma(\bar{x}, \Lambda_r(\bar{x})) = \sigma(\bar{x}, 0) = f(\bar{x}, \Lambda_r(\bar{x})) = |\delta|(\bar{x}).$$

Hence, $|\varphi|(\bar{x}) = \sigma(\bar{x}, \Lambda_r(\bar{x}))$. Now assume $P(k)$. If $\Lambda_r(\bar{x}) = k + 1 > 0$,

$$\begin{aligned} |\varphi|(\bar{x}) &= |+|(|\delta|(\bar{x}), |\varphi|(r\bar{x})) && \text{(Lemma 4.6.2)} \\ &= |+|(|\delta|(\bar{x}), \sigma(r\bar{x}, \Lambda_r(r\bar{x}))) && \text{(inductive hypothesis)} \\ &= \sigma(\bar{x}, \Lambda_r(\bar{x})) && \text{(by (iv) above)} \end{aligned}$$

Thus, for every $\bar{x} \in \mathbb{N}^n$ with $\Lambda_r(\bar{x}) \neq \bot$, we have $|\varphi|(\bar{x}) = \sigma(\bar{x}, \Lambda_r(\bar{x}))$.

Otherwise, $\Lambda_r(\bar{x}) = \bot$. Then $\bigsqcup r^n \bar{x} = \infty^n$, where this fixed point is achieved in either finite time ($\lambda_r x < \infty$), or r loops forever ($\lambda_r x = \infty$). In the finite case, it is obvious that $\varphi(\bar{x}) = \infty$, while Lemma 4.6.5 shows that $\varphi(\bar{x}) = \infty$ in the infinite case. Hence, $|\varphi|(\bar{x}) = \sigma(\bar{x}, \Lambda_r(\bar{x})) = \bot$.

Therefore, $|\varphi|$ is partial recursive, which puts $\varphi \in \mathcal{A}$, finishing the proof. $\square$

Thus, the image of the partial recursives under the bijection in Proposition 4.6.1 is exactly $\mathcal{C}$.

Theorem 4.6.1 $|\mathcal{C}|$ *is the class of partial recursive functions.*

4.7 The Primitive Recursive Functions

In this section we set $D = \mathbb{N}^*$, $\mathcal{C} = \mathcal{C}(D)$, and prove that $\mathcal{C} = |\mathcal{C}|$ is the class of primitive recursive functions on the naturals.

Definition 4.7.1 The class of *primitive recursive functions* is the smallest class of functions $f : \mathbb{N}^n \to \mathbb{N}$, for $n \geq 1$, which contains the zero function, the successor function, the projections, and is closed under composition and primitive recursion.

As we go through the proof of the case $D = \mathbb{N}^*$, it is useful to keep in mind that $\varphi = \delta + \varphi \circ r$ is a renee equation on $D^n \to D$ iff $+$ is idempotent at $\delta(x)$, for every $x \in \text{fix}(r)$. The reason for this is that all elements of $\mathbb{N}^*$ are compact.

Lemma 4.7.1 *The following arithmetic functions belong to $\mathcal{C}$:*

(i) *The constant functions* 0 *and* 1.

(ii) *The succesor* $\text{succ} : D \to D$ *given by* $\text{succ}(x) = x + 1$.

(iii) *The predecessor* $\text{pred} : D \to D$ *given by* $\text{pred}(x) = x \ominus 1$.

(iv) *The predicates* $<$, $=$, *and* $>$.

(v) *Integer division* $\div : D^2 \to D$ *given by*

$$x \div y = \begin{cases} x/y & \text{if } y \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

(vi) *The remainder function* $\text{rem} : D^2 \to D$ *for integer division given by*

$$x \operatorname{rem} y = \begin{cases} x - (x \div y) \times y & \text{if } y \neq 0; \\ x & \text{otherwise.} \end{cases}$$

proof The proof of (i), (ii), and (iv) given in Lemma 4.6.3 works here as well. For (iii), $\text{pred}(x) = x \ominus 1$, and since $\ominus$ is initial, $\text{pred} \in \mathcal{C}$ by substitution. For (v), we solve a renee equation as follows: Let

$$\delta(x, y) = (y \leq x) \times (y > 0) \text{ and } r(x, y) = (x \ominus y, y)$$

and as the iterative operation, take the usual addition $+ : D^2 \to D$. Then $\varphi = \delta + \varphi \circ r \in \mathcal{C}$ is a renee equation: The fixed points of r are

$$\{(0, y) : y \in D\} \cup \{(x, 0) : x \in D\}$$

and $+$ is idempotent at $\delta(x, 0) = \delta(0, y) = 0$. Hence, $x \div y = \varphi(x, y) \in \mathcal{C}$, by substitution. For (vi), $x \operatorname{rem} y := x \ominus (x \div y) \times y \in \mathcal{C}$, by substitution. □

Lemma 4.7.2 *The primitive recursive bijection* $b_2 : \mathbb{N}^2 \to \mathbb{N}$, *given by*

$$b_2(x, y) = 2^x(2y + 1) - 1,$$

belongs to $\mathcal{C}$. The coordinates $\pi_i^2 \circ b_2^{-1} : \mathbb{N} \to \mathbb{N}$, *for* $1 \leq i \leq 2$, *of its inverse* $b_2^{-1} : \mathbb{N} \to \mathbb{N}^2$ *also belong to $\mathcal{C}$.*

proof To prove that b_2 belongs to $\mathcal{C}$ we need only prove that $x \mapsto 2^x$ is in $\mathcal{C}$, and then appeal to substitution. For this, we solve a renee equation on $D \to D$ as follows: Let $\delta(x) = 1 \times (x = 0) + 2 \times (x > 0)$ and consider

$$\exp_2 = \delta \times \exp_2 \circ \text{pred}$$

where our iterative operation is the usual multiplication $\times : D^2 \to D$. This is a renee equation: $\times$ is idempotent at $\delta(x) = 1$, for every $x \in \text{fix}(\text{pred}) = \{0\}$. Thus, the function $2^x = \exp_2(x)$ belongs to $\mathcal{C}$, which by substitution implies that $b_2 \in \mathcal{C}$ does too.

The inverse of b_2 can be written as

$$b_2^{-1}(x) = (\pi_1(x), \pi_2(x))$$

where $\pi_1(x) =$ the exponent of 2 in the prime factorization of $x + 1$, and

$$\pi_2(x) = \frac{((x+1)/2^{\pi_1(x)}) - 1}{2}.$$

Clearly, then, it is enough to prove that π_1 is in $\mathcal{C}$, and then appeal to substitution for the case of π_2.

To prove that π_1 is in $\mathcal{C}$, we solve a renee equation on $D \to D$ as follows: Let $\delta(x) = (x \,\text{rem}\, 2 = 0) \times (x > 0)$, $r(x) = (x \,\text{rem}\, 2 = 0) \times (x \div 2)$ and consider $\varphi = \delta + \varphi \circ r$, where $+ : D^2 \to D$ is the usual addition. This is a renee equation as $+$ is idempotent at $\delta(x) = 0$, for all $x \in \text{fix}(r) = \{0\}$. Then $\pi_1 = \varphi \circ \text{succ} \in \mathcal{C}$. $\square$

Lemma 4.7.3 *For each $n \geq 1$, $b_n : \mathbb{N}^n \to \mathbb{N}$, defined inductively by*

$$b_1(x) = x$$

$$b_2(x, y) = 2^x(2y + 1) - 1$$

$$b_{n+1}(\bar{x}, y) = b_2(b_n(\bar{x}), y),$$

is a primitive recursive bijection which belongs to $\mathcal{C}$. Furthermore, for all $n \geq 1$ and $1 \leq i \leq n$, the coordinate $\pi_i^n \circ b_n^{-1} : \mathbb{N} \to \mathbb{N}^n$ also belongs to $\mathcal{C}$.

proof The lemma holds trivially for $n = 1$ so we consider the case $n \geq 2$. Observe that $b_{n+1} : \mathbb{N} \to \mathbb{N}^{n+1}$ is given by

$$b_{n+1}^{-1}(s) = (b_n^{-1} \circ \pi_1^2 \circ b_2^{-1}(s), \pi_2^2 \circ b_2^{-1}(s)),$$

for $n \geq 1$. We prove that $b_n \in \mathcal{C}$ and that the coordinates of its inverse are in $\mathcal{C}$, for all $n \geq 2$, by induction. The base case $n = 2$ was done in Lemma 4.7.2. Now assume that for $n \geq 2$, we have

$$b_n, \pi_i^n \circ b_n^{-1} \in \mathcal{C},$$

for all $1 \leq i \leq n$. Then by definition, $b_{n+1}(\bar{x}, y) = b_2(b_n(\bar{x}), y)$, but $b_2 \in \mathcal{C}$ by Lemma 4.7.2, and $b_n \in \mathcal{C}$ by the inductive hypothesis. Thus, $b_{n+1} \in \mathcal{C}$, as $\mathcal{C}$ is closed under substitution.

To see that the coordinates of b_{n+1}^{-1} are in $\mathcal{C}$, observe that

$$\pi_i^{n+1} \circ b_{n+1}^{-1} = (\pi_i^n \circ b_n^{-1}) \circ (\pi_1^2 \circ b_2^{-1}) \in \mathcal{C}$$

for $1 \leq i \leq n$, because $\pi_i^n \circ b_n^{-1} \in \mathcal{C}$ by the inductive hypothesis, $\pi_1^2 \circ b_2^{-1} \in \mathcal{C}$ by Lemma 4.7.2, and $\mathcal{C}$ is closed under substitution. Finally,

$$\pi_{n+1}^{n+1} \circ b_{n+1}^{-1} = \pi_2^2 \circ b_2^{-1} \in \mathcal{C},$$

by Lemma 4.7.2. Hence, $\pi_i^{n+1} \circ b_{n+1}^{-1} \in \mathcal{C}$, for all $1 \leq i \leq n+1$, finishing the proof. □

Proposition 4.7.1 *Every primitive recursive function belongs to $\mathcal{C}$.*

proof Because $D = \mathbb{N}$ as sets, every function in $\mathcal{C}$ is a function $f : \mathbb{N}^n \to \mathbb{N}$, for $n \geq 1$. We now show that $\mathcal{C}$ contains the initial functions and that it is closed under composition and primitive recursion.

$\mathcal{C}$ contains the initial functions: It contains the zero function, the successor function, and the projection mappings π_i^n.

$\mathcal{C}$ is closed under composition: Composition of maps on $\mathbb{N}$ is the same as substitution of maps on D in this case.

$\mathcal{C}$ is closed under primitive recursion: We solve a renee equation. If $g : \mathbb{N}^n \to \mathbb{N}$ and $h : \mathbb{N}^{n+2} \to \mathbb{N}$ are in $\mathcal{C}$, for $n \geq 1$, we want to show that $f : \mathbb{N}^{n+1} \to \mathbb{N}$ given by

$$f(\bar{x}, y) = \begin{cases} g(\bar{x}) & \text{if } y = 0; \\ h(\bar{x}, y-1, f(\bar{x}, y-1)) & \text{otherwise.} \end{cases}$$

is also in $\mathcal{C}$. For this proof, let $p(x) = \text{pred}(x)$. First we define a splitting $r : D^{n+1} \to D^{n+1}$ by

$$r(\bar{x}, y) = (\bar{x}, p(y)),$$

and use it to define a mapping $\delta : D^{n+1} \to D$ by

$$\delta(\bar{x}, y) = [b_{n+1}(r(\bar{x}, y)) + 1] \times (y > 1) + (g(\bar{x}) + 1) \times (y = 1),$$

where $b_{n+1} : \mathbb{N} \to \mathbb{N}^{n+1}$ is the bijection of Lemma 4.7.3, and a binary operation $\oplus : D^2 \to D$ by

$$x \oplus y = [h(r \circ b_{n+1}^{-1} \circ p(x), p(y)) + 1] \times (x \times y > 0) + x \times (y = 0) + y \times (x = 0).$$

We claim that $\varphi = \delta \oplus \varphi \circ r$ is a renee equation. The fixed points of r are $\text{fix}(r) = \{(\bar{x}, 0) : \bar{x} \in D^n\}$. For any $(\bar{x}, 0) \in \text{fix}(r)$, we have $\delta(\bar{x}, 0) = 0$, and since $0 \oplus 0 = 0$, δ maps the fixed points of r to idle points of $\oplus$. Hence, we have a renee equation $\varphi = \delta \oplus \varphi \circ r \in \mathcal{C}$.

Let $\bar{x} \in D^n$ be fixed. We claim that $\varphi(\bar{x}, y + 1) = f(\bar{x}, y) + 1$, for all $y \in D$. The proof is by induction. For the base case $y = 0$, we have

$$\begin{aligned}
\varphi(\bar{x}, 0 + 1) &= \delta(\bar{x}, 1) \oplus \varphi(\bar{x}, 0) \\
&= (g(\bar{x}) + 1) \oplus 0 \\
&= g(\bar{x}) + 1 \\
&= f(\bar{x}, 0) + 1.
\end{aligned}$$

Now assume the claim is true for $y \geq 0$. Then

$$\begin{aligned}
\varphi(\bar{x}, y + 2) &= \delta(\bar{x}, y + 2) \oplus \varphi(\bar{x}, y + 1) \\
&= [b_{n+1}(r(\bar{x}, y + 2)) + 1] \oplus (f(\bar{x}, y) + 1) \quad \text{(inductive hypothesis)} \\
&= [b_{n+1}(\bar{x}, y + 1) + 1] \oplus (f(\bar{x}, y) + 1) \\
&= h(r \circ b_{n+1}^{-1} \circ p(b_{n+1}(\bar{x}, y + 1) + 1), p(f(\bar{x}, y) + 1)) + 1 \\
&= h(\bar{x}, y, f(\bar{x}, y)) + 1 \\
&= f(\bar{x}, y + 1) + 1,
\end{aligned}$$

which proves the claim. Because $\bar{x}$ was arbitrary in the argument above,

$$\varphi(\bar{x}, y + 1) = f(\bar{x}, y) + 1,$$

for all $(\bar{x}, y) \in D^{n+1}$. Now define $s : D^{n+1} \to D$ by $s(\bar{x}, y) = \varphi(\bar{x}, \text{succ}(y))$. By substitution, $s \in \mathcal{C}$, and so $\text{pred} \circ s \in \mathcal{C}$. But since $s(\bar{x}, y) \geq 1$, $\text{pred}(s(\bar{x}, y)) = s(\bar{x}, y) - 1 = f(\bar{x}, y)$, for all $(\bar{x}, y) \in D^{n+1}$. Then we have $f = \text{pred} \circ s \in \mathcal{C}$.

Thus, $\mathcal{C}$ is a class of functions containing the initial functions which is closed under composition and primitive recursion. As the class of primitive recursives is the smallest such class, it must be contained in $\mathcal{C}$. $\square$

For each $n \geq 1$, D^n has a natural measurement $|\cdot| : D^n \to D$ given by

$$|(x_1, \cdots, x_n)| = x_1 + \cdots + x_n$$

such that
$\bar{x} \sqsubseteq \bar{y}$ & $|\bar{x}| = |\bar{y}| \Rightarrow \bar{x} = \bar{y}$, for all $\bar{x}, \bar{y} \in D^n$.

Lemma 4.7.4 *If $r : D^n \to D^n$ is a splitting on D^n, then*

$$r^{|\bar{x}|}(\bar{x}) \in \text{fix}(r),$$

for all $\bar{x} \in D^n$.

proof The proof is by induction on $|\bar{x}|$. For the base case $|\bar{x}| = 0$, we have

$$r^{|\bar{x}|}(\bar{x}) = r^0(\bar{x}) = \bar{x} = (0, \cdots, 0) = 0^n \in \text{fix}(r),$$

since $|\bar{x}| = 0 \Rightarrow \bar{x} = 0^n$. Now assume the claim holds for all $\bar{x} \in D^n$ with $|\bar{x}| \leq k$.

If $|\bar{x}| = k + 1$, then either $|r\bar{x}| = |\bar{x}|$ or $|r\bar{x}| < |\bar{x}|$. If $|r\bar{x}| = |\bar{x}|$, then $r(\bar{x}) = \bar{x}$, and so $r^{k+1}(\bar{x}) = \bar{x} \in \text{fix}(r)$, since $1 \leq k+1$. Otherwise, $|r\bar{x}| < |\bar{x}|$. Then $|r\bar{x}| \leq k$. By the inductive hypothesis,

$$r^{|r\bar{x}|}(r\bar{x}) = r^{|r\bar{x}|+1}(\bar{x}) \in \text{fix}(r),$$

and since $|r\bar{x}| + 1 \leq k + 1$, $r^{k+1}(\bar{x}) \in \text{fix}(r)$. $\square$

Theorem 4.7.1 *$\mathcal{C}$ is the class of primitive recursive functions.*

proof Denote the class of primitive recursive functions by $\mathcal{P}$. By Prop. 4.7.1, $\mathcal{P} \subseteq \mathcal{C}$. Because each $f : \mathbb{N}^n \to \mathbb{N} \in \mathcal{P}$ is also a map $f : D^n \to D \in \mathcal{C}$, we prove that $\mathcal{P}$ contains the initial functions and is closed under substitution and iteration.

$\mathcal{P}$ contains the initial functions: $+, \ominus, \times, \leq$ and the projections π_i^n are all primitive recursive.

$\mathcal{P}$ is closed under substitution: Composition of maps on $\mathbb{N}$ is the same as substitution of maps on D.

$\mathcal{P}$ is closed under iteration: Let $\varphi = \delta + \varphi \circ r$ be a renee equation on $D^n \to D$, for $n \geq 1$, where $\delta, +$, and the coordinates of $r : D^n \to D^n$ all belong to $\mathcal{P}$. We must prove that $\varphi \in \mathcal{P}$.

Examining the proof of Lemma 4.6.4 reveals that

$$f : \mathbb{N}^{n+1} \to \mathbb{N}$$

$$f(\bar{x}, k) = \delta(r^k(\bar{x}))$$

is primitive recursive because δ, $\pi_i^n \circ r$, b_n and $\pi_i^n \circ b_n^{-1}$ are all primitive recursive, for all $1 \leq i \leq n$, where b_n is the primitive recursive bijection of Lemma 4.7.3. Finally, from the proof of Prop. 4.6.3, recall the mapping

$$\sigma : \mathbb{N}^{n+1} \to \mathbb{N}$$

$$\sigma(\bar{x}, k) = \begin{cases} f(\bar{x}, \Lambda_r(\bar{x})) & \text{if } k = 0; \\ +(\, f(\bar{x}, \Lambda_r(\bar{x}) \ominus k), \sigma(\bar{x}, k-1)\,) & \text{if } k > 0; \end{cases}$$

where we abbreviate Λ_r to $\Lambda_r(\bar{x}) = \min\{k \in \mathbb{N} : r^k(\bar{x}) = r^{k+1}(\bar{x})\}$. We have already shown in Prop. 4.6.3 that $\varphi(\bar{x}) = \sigma(\bar{x}, \Lambda_r(\bar{x}))$, for all $\bar{x} \in \mathbb{N}^n$. However, this *does not* finish the present proof: We must show that σ and Λ_r are not only partial recursive, but in fact *primitive recursive.*

First we consider the case of $\Lambda_r : \mathbb{N}^n \to \mathbb{N}$. This function is the minimization of a primitive recursive function, and this minimization is itself *bounded above* by a primitive recursive function: That is, for all $\bar{x} \in \mathbb{N}^n$,

$$\Lambda_r(\bar{x}) \leq |\bar{x}|,$$

in view of Lemma 4.7.4, where we again emphasize that the measurement $|\cdot| : \mathbb{N}^n \to \mathbb{N}$ is primitive recursive. Then because the class of primitive recursive functions is closed under minimization with a primitive recursive bound (see [4], p.87, for example), we have that $\Lambda_r \in \mathcal{P}$.

The case of σ is now trivial: It is defined by primitive recursion using only primitive recursive functions.

Then by composition, $\varphi(\cdot) = \sigma(\cdot, \Lambda_r(\cdot)) \in \mathcal{P}$, which proves that $\mathcal{P}$, in addition to containing the intial functions and being closed under substitution, is also closed under iteration. As $\mathcal{C}$ is the smallest such class, we have $\mathcal{C} \subseteq \mathcal{P}$, which finishes the proof. □

4.8 The Flat Recursive Functions

In this section we set $D = \mathbb{N}^\flat$, $\mathcal{C} = \mathcal{C}(D)$, and prove that $\mathcal{C} = |\mathcal{C}|$ is the smallest class of functions on the naturals containing the intitial functions which is closed under composition.

Theorem 4.8.1 *$\mathcal{C}$ is the smallest class of functions $f : \mathbb{N}^n \to \mathbb{N}$, for $n \geq 1$, such that*

(i) *$\mathcal{C}$ contains $+$, $\ominus$, $\times$, $\leq$, and π_i^n, for $n \geq 1$ and $1 \leq i \leq n$,*

(ii) *$\mathcal{C}$ is closed under composition: If $f : \mathbb{N}^n \to \mathbb{N}$ is in $\mathcal{C}$ and $g_i : \mathbb{N}^k \to \mathbb{N}$ is in $\mathcal{C}$, for $1 \leq i \leq n$, then*

$$f(g_1, \cdots, g_n) : \mathbb{N}^k \to \mathbb{N} \in \mathcal{C}.$$

proof Let $\mathcal{A}$ be the smallest class of maps satisfying (i) and (ii) above.

Because $D = \mathbb{N}$ as sets, the members of $\mathcal{C}$ are functions $f : \mathbb{N}^n \to \mathbb{N}$, for $n \geq 1$. With this understanding, we see that $\mathcal{C}$ satisfies (i) and (ii) by definition, so $\mathcal{A} \subseteq \mathcal{C}$.

For the other inclusion, we notice that the functions in $\mathcal{A}$ are mappings $f : D^n \to D$, for $n \geq 1$, and then prove that $\mathcal{A}$ is closed under iteration. Let $\varphi = \delta + \varphi \circ r$ be a renee equation on $D^n \to D$, where $\delta : D^n \to D$ and $+ : D^2 \to D$ are in $\mathcal{A}$, and $r : D^n \to D^n$ is a map whose coordinates are in $\mathcal{A}$. Because the order on D^n is flat, $r(x) = x$, for all $x \in D^n$. By Theorem 4.3.1, $\varphi(x) = \delta(x)$, for all $x \in \text{fix}(r)$. But $\text{fix}(r) = D^n$, so $\varphi = \delta \in \mathcal{A}$. Then $\mathcal{A}$ is closed under iteration and hence $\mathcal{C} \subseteq \mathcal{A}$. □

4.9 Computability and the Information Order

Let the symbol $\Rightarrow$ represent the word "determines," and consider the following sequence of implications:

the information order $\Rightarrow$ renee equations $\Rightarrow$ iteration $\Rightarrow$ computability.

The first of these is clear since all of the renee equation is founded upon fundamentals of the μ topology, which of course is derived from the approximation relation.

The second of these, namely, that the renee equation determines our ability to iterate could be doubted, but its assertion is certainly reasonable: One may capture the partial recursives using Scott continuous semigroup operations, while the renee equation grants us the freedom of a μ continuous operation locally associative around certain idle points.

The last of these, that the ability to iterate, *taken with basic operations and composition,* determines what we may compute, is based on the author's experience as a computer scientist.

However, when one combines all three of these they end up with the assertion that the information order on a domain determines a natural notion of computability. That is, the information order determines what the algorithms on a domain are. Put yet another way, computability is essentially an intrinsic property carried by domains. This one is not as obvious. To illustrate, the notion of computability determined by the information order on $\mathbb{N}^{\infty}$ is that of the partial recursives. That is, according to the information order, a mapping on $\mathbb{N}^{\infty}$ is an algorithm iff it is partial recursive. As more evidence, we offer the following summary.

Theorem 4.9.1

(i) $|\mathcal{C}(\mathbb{N}^{\infty})|$ *is the class of partial recursive functions.*

(ii) $|\mathcal{C}(\mathbb{N}^{*})|$ *is the class of primitive recursive functions.*

(iii) $|\mathcal{C}(\mathbb{N}^{\flat})|$ *is the smallest class of functions containing the initial functions which is closed under composition.*

Using the idea that, ultimately, it is the order on a domain that determines computability, it is very easy to see why the previous result is true.

On $\mathbb{N}^{\infty}$, a splitting can loop *until* a certain condition is met. It may, for instance, loop forever. This has the same computational power as does programming with while loops, and so we obtain the partial recursives.

In the case of $\mathbb{N}^{*}$, the number of times a splitting $r : \mathbb{N}^n \to \mathbb{N}^n$ can loop on input $\bar{x} = (x_1, \cdots, x_n)$ is bounded by $|\bar{x}| = x_1 + \cdots + x_n$. Thus, the number of steps that an algorithm $\varphi = \delta + \varphi \circ r$ may use to compute its output is *predetermined* by the measure $|\bar{x}|$ of its input $\bar{x}$ – just like a counting loop. Hence we obtain the primitive recursives.

The case of $\mathbb{N}^{\flat}$ is even easier: The only splitting on a flatly ordered set is the identity map, so one cannot iterate at all.

Then what we see is not in question: The information order on a domain determines a notion of computability *because* it determines our ability to iterate. Indeed, what else accounts for the change we see before us? We perform the exact same construction on the set of natural numbers in all cases: The same initial functions, the same closure under substitution, the same closure under iteration. The only thing that changes in each case is the information order. And it is because of this *one difference,* that we arrive at three very different notions of computability.

4.10 The Influence of Measurement

Everything done here rests on the μ topology which we know essentially is a sort of abstract measurement in the form of the identity map. However, if that were the only link to measurement, measurement would have to be a useless and random idea.

So suppose we have an algorithm

$$\varphi = \delta + \varphi \circ r$$

with $\lambda_r x = \infty$ for an input x. Then φ loops infinitely on input x.

For example, if $\varphi \in \mathcal{C}(\mathbb{N}^\infty)$, that is, if φ is partial recursive, we know $\varphi(x) = \infty$. Put another way,

$$\varphi(x) \in \ker \mu,$$

where $\mu x = 1/2^x$ is the standard measurement on $\mathbb{N}^\infty$. To this one naturally says "So what?" The idea being that no matter how we phrase it, ∞ is still useless. But now consider another infinite loop:

$$\varphi = 1_{\mathbf{I}\mathbb{R}} \sqcup \varphi \circ \text{split}_f$$

The value of this function at an interval $[a, b] \in C(f)$ is now in the kernel of the length measurement, but its value

$$\varphi(x) = \bigsqcup_{n \geq 0} \text{split}_f^n[a, b] = r$$

is definitely not useless: It is a solution of the equation $f(x) = 0$, and the fact that we have an infinite loop, in this case, is merely a reflection of the fact that we have a process capable of *approximating* an ideal element r up to arbitrarily high levels of accuracy.

Now if we return to the case of a partial recursive φ with $\varphi(x) = \infty$ we see that this is exactly what an infinite loop is really doing: Attempting to approximate the ideal value ∞. The reason the idea makes no sense in this case is that the Scott topology on the kernel $\ker \mu = \{\infty\}$ is discrete. That is, since the only open subset of the kernel around ∞ is $\{\infty\}$, the only way to approximate ∞ is to compute it exactly: Something we cannot do, because φ only terminates if $\lambda_r x < \infty$. This highlights for us a natural law of computation: Discreteness is the absence of approximation.

What determines the usefulness of an infinite loop is the kernel of a measurement. If the kernel has sufficient topological structure, for example, like a copy of the real line, then we expect that infinite loops define functions worth approximating. On the other hand, if its topological structure is uninteresting, as is the case with the partial recursives, where it is simply a one point space $\{\infty\}$, then we expect that functions defined by infinite loops will not be worth approximating.

But much more is true of measurements: They not only determine whether or not a function defined by an infinite loop is worth approximating, via the structure of the kernel, they also enable us to say in precise terms *what it means* to approximate such a function.

4.11 Closing Remarks

There are numerous directions for future work. One promising direction is to utilize the ideas developed here to discuss computability on domains. A natural way to do this is to begin with a domain D and a finite set $\mathcal{I}$ of functions of the form $f : D^n \to D$, called initial functions, which are known *a priori* to be computable, and then to study the smallest class $\mathcal{C}(D)$ of functions containing $\mathcal{I}$ and the projections, which is closed under substitution and iteration. The author believes that such an approach may free us from having to distinguish between continuous and discrete notions of computability.

The general trend, for example, in developing notions of computability on domains, is to begin by assuming the partial recursives, and then to build on top of them. Of course, if this is the approach one is going to take, they might as well develop notions of computability anywhere. The question is "Why develop notions of computability for domains?" In this paper, we have answered this question in a simple and explicit manner by proving, among other things, that the partial recursive functions are *a consequence* of the domain-theoretic structure of $\mathbb{N}^\infty$. Our answer to the question might then be: One does not develop notions of computability for domains, *notions of computability exist because there are domains.* This elevates domain theory from the rank of a "chosen" or "preferable" theory to a *necessary* one – I know of no other class of mathematical objects whose intrinsic structure determines computability as demonstrated here.

The removal of adjectives like "continuous" and "discrete" from everyday scientific discourse is the moral obligation of every mathematician. The present work strongly suggests that the domain theorist is in the unique position of being able to realize this ideal within the realm of computation.

4.12 Questions

(i) If we think of μ-σ maps on domains as being analogous to the continuous mappings from analysis, find the domain theoretic idea which most closely parallels the *measurable mappings* of analysis. Is there a fixed point theorem for such mappings? Is the bisection split_f measurable? For example, consider split_f for $f(x) = x$ as a map $\text{split}_f : \downarrow[0] \to \downarrow[0]$. Is this function measurable as a map from the Borel sets of $\downarrow[0]$'s μ topology to the Borel sets of $\downarrow[0]$'s Scott topology?

(ii) What conditions on a mapping $f : D \to D$ other than monotonicity ensure that $(f^n(x))$ is an increasing sequence given $x \sqsubseteq f(x)$? More generally, that $(f^n(x))$ has a μ limit?

(iii) Splittings with continuous measure, μ-σ's with monotone measure, μ continuous maps and Scott continuous maps all have the property that

$$(x_n) \textit{ increasing } \& \ (f(x_n)) \textit{ increasing} \Rightarrow f(\bigsqcup x_n) = \bigsqcup f(x_n).$$

Is this a form of continuity? That is, is there a topology τ on a domain such that a selfmap is continuous with respect to τ iff it has the property cited above?

(iv) When is a mapping $f : D \to E$ inductive?

(v) Why are Scott domains with inf operations idle everywhere important?

(vi) The equation $\varphi = \delta + \varphi \circ r$ can be shown to have a solution without assuming $+$ is locally associative. What is a proper notion of canonicity in this case?

(vii) An interesting setting for infinite sums is the following. Let $(D, +, \mu)$ be a domain with a μ continuous operation and a measurement μ. For every $x \in \ker\mu$, suppose there is a μ open semigroup $\sigma(x)$ such that if (x_n) is a sequence in $\sigma(x)$ with $x_n \to x$ in the μ topology and

$$\sum_{n\geq 0} \mu x_n < \infty,$$

then

$$\sum_{n\geq 0} x_n := \lim_{n\to\infty} (x_1 + \cdots + x_n)$$

exists in the *Lawson* topology, and

$$\lim_{n\to\infty} \sum_{k\geq n} x_k = \lim_{n\to\infty} x_n$$

as a limit in the μ topology. Any iterative operation on a domain with a measurement whose kernel is idle qualifies as an example of this sort. In addition, however, we are now able to consider examples like $([0,\infty)^*, +, 1_{[0,\infty)^*})$. On the other hand, equations like $\varphi = \delta + \varphi \circ r$ may no longer have unique solutions. In how general of a situation can we solve equations canonically?

(viii) If $+ : D^2 \to D$ is iterative, when does it have a *basis* of semigroups at an idle point?

(ix) Given $\delta, +, r$, order $\Delta = \{f : f = \delta \text{ on fix}(r) \;\&\; f \text{ varies with } r\}$ so that it becomes a domain with a measurement μ and least element δ. Now show that the function

$$\phi : \Delta \to \Delta$$

$$\phi(f) = \delta + f \circ r$$

is a contraction: That is, show that it is monotone and that $\exists r < 1$ with $\mu\phi \leq r \cdot \mu$. Then basic results on measurement show that such a map has a unique maximal fixed point. This fixed point is also the solution of the renee equation. This provides a clear illustration of the difference between the classical domain theoretic view of algorithm and what we have done in this chapter.

Chapter 5

The Kernel as a Space

Measurements provide a degree of approximation for the members of its kernel: The elements with measure zero. The members of the kernel represent a computational ideal we hope to approximate, and a measurement tells us how close a given approximation is. In this chapter we will consider the kernel of a measurement from the topological viewpoint. In doing so, we shall gain valuable intuition about the structure of the objects we compute.

5.1 Motivation

Let us return to the question raised after Example 2.6.1. Recall that given a contraction $f : X \to X$ on a complete metric space X, we extended it to a monotone map $\bar{f} : \mathbf{B}X \to \mathbf{B}X$ by $\bar{f}(x, r) = (fx, c \cdot r)$, where $c < 1$ is the Lipschitz constant for f. We then applied Proposition 2.6.2 to $\bar{f}$ and concluded that f had a unique fixed point $x^\star$. The question is, How do we establish that $x^\star$ is an attractor?

Theorem 5.1.1 *Let D be a domain with a measurement μ such that*

$$(\,\forall\, x, y \in \ker \mu\,)(\,\exists\, z \in D\,)\; z \sqsubseteq x, y.$$

If $f : D \to D$ is a monotone map for which there is a constant $c < 1$ such that $\mu f(x) \leq c \cdot \mu x$, for all $x \in D$, and there is a point $x \in D$ with $x \sqsubseteq f(x)$, then

$$x^\star = \bigsqcup_{n \geq 0} f^n(x) \in \max D$$

is the unique fixed point of f on D. Furthermore, $x^\star$ is an attractor: For all $x \in \ker \mu$, $f^n(x) \to x^\star$ in the Scott topology on $\ker \mu$.

proof To see that $x^\star$ is an attractor, let $x \in \ker \mu$. Then $f^n(x) \in \ker \mu$ for all $n \geq 0$. In addition, there is an $a \sqsubseteq x, x^\star$, so $f^n(a) \sqsubseteq f^n(x), x^\star$. Now let U be a Scott open set around x^*. Because μ is a measurement,

$$(\exists\, \varepsilon > 0)\; x^\star \in \mu_\varepsilon(x^\star) \subseteq U.$$

Since $\mu f^n(a) \to 0$, all but a finite number of the $f^n(a)$ are in U. But U is an upper set, so the same is true of the $f^n(x)$. Hence, $f^n(x) \to x^\star$, in the Scott topology on $\ker \mu$. All else follows from Proposition 2.6.2. $\Box$

To prove that $x^\star$ is an attractor, let $x \in X$ be arbitrary. By Theorem 5.1.1, $\bar{f}^n(x, 0) = (f^n x, 0) \to (x^\star, 0)$ in the relative Scott topology on $\ker \pi$. However, as a space in its relative Scott topology, $\ker \pi \simeq X$, with an explicit homeomorphism given by $(x, 0) \mapsto x$. Hence, $f^n x \to x^\star$ in X. Thus, it is precisely because $\ker \pi = \max \mathbf{B}X \simeq X$ that Theorem 5.1.1 implies $x^\star$ is an attractor for f.

Another motivation for studying the topological structure of the kernel was encountered at the end of Chapter 4, where we saw that it can determine whether or not infinite loops (in renee equations) define functions worth approximating. The subject of this chapter, then, concerns the types of spaces that may be represented as the kernel of a measurement μ.

5.2 Topological Background

We will need to use some results from topology which do not appear in standard texts, so we give them here.

Definition 5.2.1 A *symmetric* on a space X is a function $d : X^2 \to [0, \infty)$ such that for all $x, y \in X$,

(i) $d(x, y) = d(y, x)$ and

(ii) $d(x, y) = 0 \Leftrightarrow x = y$.

A symmetric places a topology on X where a set $U \subseteq X$ is open iff

$$(\forall x \in U)(\exists\, \varepsilon > 0)\; x \in B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\} \subseteq U.$$

A space is *symmetrizable* if its topology is given by a symmetric.

The following result appears in [9](p.137), [2](p.30), and [3].

Theorem 5.2.1 *If (X, d) is a symmetrizable Hausdorff space and*

$$d(x, x_n) \to 0, d(x_n, y_n) \to 0 \Rightarrow d(x, y_n) \to 0,$$

then X is metrizable.

The next lemma is due to Frink [9] and is treated nicely in [11].

Lemma 5.2.1 (Frink's Lemma) *If $d : X^2 \to [0, \infty)$ is a function such that*

$$(\forall \varepsilon > 0)\, d(x, y) < \varepsilon \;\&\; d(y, z) < \varepsilon \Rightarrow d(x, z) < 2\varepsilon.$$

Then $\exists\, \rho : X^2 \to [0, \infty)$ such that for all $x, y, z \in X$,

(i) $\rho(x, z) \le \rho(x, y) + \rho(y, z)$

(ii) $d(x, y)/4 \le \rho(x, y) \le d(x, y)$.

Finally, ρ is symmetric provided d is.

The following results can all be found in [8].

Definition 5.2.2 A topological space X is *Tychonoff* if it is a subspace of a compact Hausdorff space.

Tychonoff spaces are sometimes also called *completely regular*. They are the spaces which have compactifications. The largest of all such compactifications βX is known as the Stone-Čech compactification.

Definition 5.2.3 A topological space X is *Čech-complete* if it is Tychonoff and a G_δ in its Stone-Čech compactification βX.

A locally compact Hausdorff space is Čech-complete since it is an open subset of a compact Hausdorff space. Complete metric spaces provide another example.

Theorem 5.2.2 *A topological space is completely metrizable iff it is a Čech-complete metric space.*

More generally, there are the following characterizations of Čech-completeness.

Theorem 5.2.3 *A Tychonoff space X is Čech-complete iff there is a countable family $\{\mathcal{R}_i\}_{i=1}^{\infty}$ of open covers of X with the following property: For any family $\mathcal{F}$ of closed sets such that*

(i) *$\mathcal{F}$ has the finite intersection property, and*

(ii) $(\forall i)(\exists F_i \in \mathcal{F})(\exists U_i \in \mathcal{R}_i)\ F_i \subseteq U_i$

we have $\bigcap \mathcal{F} \neq \emptyset$.

Theorem 5.2.4 (Willard [19],p.183) *A Tychonoff space X is Čech-complete iff it is a G_δ in every Tychonoff space in which it is densely embedded.*

Definition 5.2.4 A sequence of open covers $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of a space X is called a *development* provided that $\{\mathrm{St}(x,\mathcal{U}_n) : n \geq 0\}$ is a basis at x where

$$\mathrm{St}(x,\mathcal{U}_n) = \bigcup\{A : x \in A \in \mathcal{U}_n\}.$$

A space with a development is termed *developable.* A *Moore space* is a regular space with a development.

5.3 The kernel as a Topological Space

The kernel of a measurement is a collection of maximal elements, which when given its inherited Scott topology from P, becomes a very interesting space to study. Anytime we make topological statements about $\ker \mu$, it will always be wth respect to the Scott topology.

Proposition 5.3.1 *If $\mu : P \to [0,\infty)^*$ is monotone and $\downarrow x \cap \downarrow y \neq \emptyset$ for all $x,y \in P$, then*

$$d : P^2 \to [0,\infty)^*$$

$$d(x,y) = \inf\{\mu z : z \ll x, y\}$$

is a Scott continuous map on P^2 below μ.

proof The order $\sqsubseteq$ on $[0,\infty)^*$ is the one opposite to the usual order $\leq$ on $[0,\infty)$, so we have $x \sqsubseteq y \Leftrightarrow x \geq y$. We give the proof so as to demonstrate that it is a general domain theoretic fact which holds for any domain (not just $[0,\infty)^*$). Then $\bigsqcup$ is written in place of inf, $\sqsubseteq$ is written for $\geq$, etc. The phrase "below" is a domain theoretic statement which means $d(x,x) \sqsubseteq \mu x$ for all $x \in P$. The monotonicity of d is clear, so for a directed set $S \subseteq P^2$ whose supremum exists,

$$\bigsqcup d(S) \sqsubseteq d(\bigsqcup S).$$

Recall that the product of continuous posets is a continuous poset in the usual way: $(x, y) \ll (a, b) \Leftrightarrow x \ll a \;\&\; y \ll b$. Now let $S \subseteq P^2$ be a directed set which has a supremum and write $(a, b) = \bigsqcup S$. If $z \ll a, b$, then $(z, z) \ll (a, b) = \bigsqcup S$, so by interpolation, $(\exists\, (c, d) \in S)\, (z, z) \ll (c, d)$. Then $\mu z \sqsubseteq \bigsqcup\{\mu x : x \ll c, d\} = d(c, d) \sqsubseteq \bigsqcup d(S)$. Then $\bigsqcup d(S)$ is an upperbound for $\{\mu z : z \ll a, b\}$, so

$$d(\bigsqcup S) = d(a, b) = \bigsqcup\{\mu z : z \ll a, b\} \sqsubseteq \bigsqcup d(S).$$

This finishes the proof that d is Scott continuous. $\Box$

Whenever a measurement is around, we shall write d for the extension to P^2 defined above. Sometimes the notation $d = d(\mu)$ is helpful in this regard.

Example 5.3.1 Consider the interval domain $\mathbf{I}\mathbb{R}$ with its standard measurement μ. Then

$$(\forall\, [a], [b] \in \max \mathbf{I}\mathbb{R})\ d([a], [b]) = |b - a|$$

This is a good example to keep in mind as we proceed.

An important aspect of $d(\mu)$ is that it detects continuity of μ.

Proposition 5.3.2 *If $\mu : P \to [0, \infty)^*$ is monotone and $d = d(\mu)$ is defined, then the following are equivalent:*

(i) *μ is continuous.*

(ii) *$\mu x = d(x, x)$.*

(iii) *$x \sqsubseteq y \;\Rightarrow d(x, y) = \mu x$.*

proof (i) $\Rightarrow$ (ii): By continuity of μ, $\mu x = \bigsqcup_{a \ll x} \mu a = d(x, x)$. (ii) $\Rightarrow$ (iii): If $x \sqsubseteq y$, then $\{z \in P : z \ll x, y\} = \{z \in P : z \ll x\}$. (iii) $\Rightarrow$ (i): Applying (iii) with $y = x$, we see that $\mu x = \bigsqcup_{a \ll x} \mu a$, which implies continuity of μ. $\Box$

Definition 5.3.1 *For $\mu : P \to [0, \infty)^*$ monotone and $d = d(\mu)$ defined, we set*

$$B_\varepsilon(x) := \{\, y \in P : d(x, y) < \varepsilon \,\}$$

for all $x \in P$, $\varepsilon > 0$.

The mapping $d = d(\mu)$ has a striking relationship to μ.

Theorem 5.3.1 *If $\mu : P \to [0,\infty)^*$ is continuous, then for all $x \in P$, $\varepsilon > 0$,*

$$B_\varepsilon(x) \;=\; \uparrow\mu_\varepsilon(x)$$

provided that $d = d(\mu)$ is defined.

proof This result also holds in full generality if phrases like "$\mu y < \varepsilon$" are replaced with ones like "$\varepsilon \ll \mu y$", etc. First let $z \in \uparrow\mu_\varepsilon(x)$. Then $\exists y \in \mu_\varepsilon(x)$ such that $y \sqsubseteq z, x$ & $\mu y < \varepsilon$. By interpolation in $[0,\infty)^*$ and continuity of μ, $(\exists\, a \ll y)\, \mu a < \varepsilon$. Then $d(x,z) \le \mu a < \varepsilon$ since $a \ll y \sqsubseteq x, z$. This proves $z \in B_\varepsilon(x)$. For the other direction, let $y \in B_\varepsilon(x)$. Then $d(x,y) < \varepsilon$. By interpolation in $[0,\infty)^*$ and the definition of d, $(\exists\, z \ll x, y)\, \mu z < \varepsilon$. Then $z \in \mu_\varepsilon(x)$ so $y \in \uparrow\mu_\varepsilon(x)$. $\square$

The mapping $d = d(\mu)$ is usually defined because in most cases any pair of elements is bounded from below. However, this is not always the case. Luckily, such a problem can be fixed with ease. Our next lemma addresses this issue.

Lemma 5.3.1 *If $\mu : P \to [0,\infty)^*$ is continuous and $\mu \to \sigma_X$, then*

$$\mu_\perp : P_\perp \to [0,\infty)^*$$

$$\mu_\perp x = \begin{cases} \frac{\mu x}{1+\mu x} & \textit{if } x \in P \\ 1 & \textit{if } x = \perp \end{cases}$$

is Scott continuous, $\ker\mu = \ker\mu_\perp$ *and* $\mu_\perp \to \sigma_X$.

proof The mapping

$$\phi : [0,\infty)^* \to [0,1)^*$$

$$t \mapsto \frac{t}{t+1}$$

is an order isomporphism which is sublinear: $\phi(x+y) \le \phi(x) + \phi(y)$. In particular, it is a measurement which induces the Scott topology on $[0,\infty)^*$ everywhere. Then the composition $\phi \circ \mu : P \to [0,1)^*$ is a measurement with $\phi \circ \mu \to \sigma_X$. Now it is easy to see that it extends to $P_\perp$. Sublinearity guarantees that other properties we will see later are also inherited. $\square$

The last lemma tells us that we may always assume a given measurement is one for which $d = d(\mu)$ is defined. The simplest way to do this is to consider $\mu_\perp$ on $P_\perp$. However, it is also possible to define a map just like $d(\mu)$ on P *without* adding a bottom. First, scale the measurement μ as in the lemma above to obtain $\lambda : P \to [0,1)^*$ where $\lambda = \phi \circ \mu$. Then the mapping

$$d : P^2 \to [0,\infty)^*$$

$$d(x,y) = \begin{cases} \inf\{\lambda z : z \ll x, y \;\&\; z \in P\} & \text{if } \downarrow x \cap \downarrow y \cap P \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

is Scott continuous as the restriction of $d(\mu_\perp)$ to P^2. Furthermore, for $x \in P$ and $\varepsilon \leq 1$, the equality $B_\varepsilon(x) = \{y \in P : d(x,y) < \varepsilon\} = \uparrow\lambda_\varepsilon(x)$ still holds.

Proposition 5.3.3 *If $\mu : P \to [0,\infty)^*$ is a measurement on a continuous poset P, then* ker μ *is a symmetrizable, first countable T_1 space with a G_δ diagonal.*

proof Since $\ker\mu \subseteq \max P$, $\ker\mu$ is T_1 in its relative Scott topology. Because $[0,\infty)^*$ is first countable at 0 and $\mu \to \sigma_{\ker\mu}$, the poset P is first countable at each point of $\ker\mu$, and so this is a property which is inherited by the subspace $\ker\mu$. Now suppose that $d = d(\mu)$ is defined. It is clear that $d(x,y) = d(y,x)$ for all $x, y \in P$ and that

$$\{B_\varepsilon(x) \cap \ker\mu : x \in \ker\mu, \varepsilon > 0\}$$

is a basis for the Scott topology on $\ker\mu$ since $B_\varepsilon(x) = \uparrow\mu_\varepsilon(x)$. The continuity of μ gives $d(x,x) = \mu x = 0$ if $x \in \ker\mu$. If $d(x,y) = 0$, then

$$(\forall\, n \geq 1)(\exists\, x_n \ll x, y)\; \mu x_n < 1/n.$$

The monotonicity of μ implies that $x, y \in \ker\mu$. Since (x_n) is a sequence of approximations of x with $\lim \mu x_n = 0$, (x_n) is directed with supremum x. But by the very same reasoning, $\bigsqcup x_n = y$, and so $x = y \in \ker\mu$. Then $d(x,y) = 0 \Leftrightarrow x = y \in \ker\mu$, which means that d when restricted to $\ker\mu \times \ker\mu$ is a symmetric which yields the Scott topology on $\ker\mu$. This proves the symmetrizability of the kernel. To see that $\ker\mu$ has a G_δ diagonal, note that the continuity of d implies $\{(x,x) : x \in \ker\mu\} = d^{-1}(0)$ is a G_δ in P^2, and hence in $\ker\mu \times \ker\mu$. Finally, if $d(\mu) = d$ is not definable on P, we can add a $\perp$ to P, and the proof just given now applies to $P_\perp$ with the measurement $\mu_\perp : P_\perp \to [0,\infty)^*$. But since $\ker\mu$ and $\ker\mu_\perp$ are the very same spaces, this finishes the proof. $\Box$

Note that in general symmetrizable spaces need not be first countable (since $B_\varepsilon(x)$ need not be open) or have G_δ diagonals.

Corollary 5.3.1 *Let $\mu : P \to [0,\infty)^*$ be a measurement on a continuous poset and $d = d(\mu)$ be the natural symmetric induced by μ. For a sequence (x_n) in $\ker \mu$ and any $x \in \ker \mu$, the following are equivalent:*

(i) $x_n \to x$.

(ii) $d(x_n, x) \to 0$.

proof This holds for any symmetric but we will do the proof specific to our setting. (i) $\Rightarrow$ (ii): Let $\varepsilon > 0$ be given. Choose $z \ll x$ with $\mu z < \varepsilon$. Then $(\exists K)\, n \geq K \Rightarrow z \ll x_n$. Thus $d(x_n, x) \leq \mu z < \varepsilon$ for $n \geq K$. (ii) $\Rightarrow$ (i): Let U be a Scott open set in P which contains x. First, $(\exists \varepsilon > 0)\, x \in \mu_\varepsilon(x) \subseteq U$. By (ii), $(\exists K)\, d(x_n, x) < \varepsilon$ for $n \geq K$. By continuity of d, $(\forall\, n \geq K)(\exists\, a_n \ll x_n, x)\, \mu a_n < \varepsilon$. Then $(\forall\, n \geq K)\, a_n \in \mu_\varepsilon(x) \subseteq U$. But U is an upper set so $x_n \in U$ for $n \geq K$. $\Box$

Proposition 5.3.4 *If $\mu : P \to [0,\infty)^*$ is a measurement on a continuous poset P, then every closed subset of $\ker \mu$ is a G_δ subset of P.*

proof Let $X \subseteq \ker \mu$ be a closed set. For each $x \in X$, there is an increasing sequence (a_n^x) with $a_n^x \ll x, \mu a_n^x < \frac{1}{n}$ & $\bigsqcup a_n^x = x$. For each $n \geq 1$, we set

$$U_n = \bigcup_{x \in X} \twoheaduparrow a_n^x.$$

This is a Scott open set in P containing X so $\bigcap U_n$ is a G_δ in P containing X. Now let $x \in \bigcap U_n$. Then $(\forall n)(\exists b_n)\, \mu b_n < \frac{1}{n}$ & $b_n \ll x$. Since $\mu b_n \to \mu x = 0$, the sequence (b_n) is directed with supremum x. Using the measurement μ we choose an increasing subsequence (b_{n_k}) of (b_n) which also has supremum x. By the definition of the U_n, $(\ \exists\, m_k \in X\ )\, b_{n_k} \sqsubseteq m_k$. The sequence (b_{n_k}) is increasing so it is easy to see that $m_k \to x$. But X is closed and $m_k \in X$ so $x \in X$. This shows that X is a G_δ subset of P. $\Box$

Proposition 5.3.5 *The kernel of a measurement on a continuous poset is developable.*

proof Let $X = \ker \mu$ and $d = d(\mu)$ be the induced symmetric on X. First, given $\varepsilon > 0$ and $x \in X$, there is a $\delta(x) > 0$ such that

$$(\forall a, b \in X)\, d(a, x) < \delta(x) \;\&\; d(x, b) < \delta(x) \Rightarrow d(a, b) < \varepsilon.$$

If not, one can find two sequences (a_n), (b_n) and $\varepsilon > 0$ such that $a_n \to x$, $b_n \to x$ and $d(a_n, b_n) \geq \varepsilon$ for all n. Now choose $z \ll x$ with $\mu z < \varepsilon$ using the continuity of μ. Since $a_n \to x$ and $b_n \to x$, $(\exists\, K)\, n \geq K \Rightarrow z \ll a_n, b_n$. This gives $d(a_n, b_n) \leq \mu z < \varepsilon$, which is a contradiction. Thus, for each $x \in X$ and $n \geq 0$, there is a $\delta_n(x) > 0$ such that

$$(\forall a, b \in X)\, d(a, x) < \delta_n(x) \;\&\; d(x, b) < \delta_n(x) \Rightarrow d(a, b) < \frac{1}{2^n}.$$

Now define an open cover by

$$\mathcal{U}_n = \{B_{\delta_n(x)}(x) \cap X : x \in X\}$$

for each $n \geq 0$. We are going to prove that the family of open covers $\{\mathcal{U}_n\}_{n=0}^{\infty}$ is a development for X. To do so we must show that

$$\{\mathrm{St}(x, \mathcal{U}_n) : n \geq 0\}$$

is a basis at x where

$$\mathrm{St}(x, \mathcal{U}_n) = \bigcup \{A : x \in A \in \mathcal{U}_n\}.$$

To see this let $x \in X$ and let U be a Scott open set in P around x. Using the fact that μ is a measurement, $(\exists\, \varepsilon > 0)\, x \in \mu_\varepsilon(x) \subseteq U$. Now choose k with $\frac{1}{2^k} < \varepsilon$. We claim that $x \in \mathrm{St}(x, \mathcal{U}_k) \subseteq U$. If $y \in \mathrm{St}(x, \mathcal{U}_k)$, then

$$(\exists\, a \in X)\, x, y \in B_{\delta_k(a)}(a).$$

But $d(x, a) < \delta_k(a) \;\&\; d(a, y) < \delta_k(a) \Rightarrow d(x, y) < \frac{1}{2^k}$, which proves

$$x \in \mathrm{St}(x, \mathcal{U}_k) \subseteq B_\varepsilon(x) = \uparrow \mu_\varepsilon(x) \subseteq U.$$

Then $X = \ker \mu$ has a development. □

Corollary 5.3.2 *The kernel of a measurement on a continuous poset is regular iff it is a Moore space.*

The next example suggests that there is not much else we can prove at the level of a continuous poset.

Example 5.3.2 Let $D = \{a_i : i \in \mathbb{N}\} \cup \{b_i : i \in \mathbb{N}\} \cup \{m_i : i \in \mathbb{N}\} \cup \{a, b, \bot\}$ be the ω-algebraic dcpo ordered by requiring (a_i) and (b_i) to be increasing sequences with suprema a and b, respectively, and that $a_i, b_i \sqsubseteq m_i$ for all $i \in \mathbb{N}$. Define a measurement on D by

$$\mu : D \to [0, \infty)^*$$

$$\mu x = \begin{cases} \frac{1}{i} & \text{if } x = a_i \text{ or } x = b_i \\ 0 & \text{if } x = m_i \text{ or } x \in \{a, b\} \\ 2 & \text{if } x = \bot \end{cases}$$

Then μ induces the Scott topology everywhere on D and $\ker \mu = \max D$, but the kernel is not even Hausdorff since the sequence (m_i) has two limits: a and b.

5.4 Lebesgue Measurements

By induction, a Scott continuous map $\mu : P \to [0, \infty)^*$ is a measurement iff $(\forall$ *finite* $F \subseteq \ker \mu)(\forall U \in \sigma_P)$,

$$F \subseteq U \Rightarrow (\exists \lambda > 0)(\forall x \in F)\ \mu_\lambda(x) \subseteq U.$$

If we require this to hold, not only of finite sets F, but of *all* compact sets K, we have exactly a Lebesgue measurement.

Definition 5.4.1 A *Lebesgue measurement* $\mu : P \to [0, \infty)^*$ is a Scott continuous map such that $(\forall$ *compact* $K \subseteq \ker \mu)(\forall U \in \sigma_P)$,

$$K \subseteq U \Rightarrow (\exists \lambda > 0)(\forall k \in K)\ \mu_\lambda(k) \subseteq U.$$

We often say that Lebesgue measurements induce the Scott topology near *compact* subsets of the kernel. The non-Hausdorff example in the last section provides an example of a measurement which is not a Lebesgue measurement. Also observe that the induced symmetric $d(\mu)$ has the Lebesgue covering property from topology *exactly* when μ is a Lebesgue measurement (hence the name).

Theorem 5.4.1 *If* $\mu : P \to [0, \infty)^*$ *is a Lebesgue measurement, then* $\ker \mu$ *is metrizable in its inherited Scott topology.*

proof Without loss of generality we can assume that the mapping

$$d : P^2 \to [0, \infty)^*$$

$$d(x, y) = \inf\{\mu z : z \ll x, y\}$$

is defined as explained earlier. By the work of the last section, restricting the function $d : P^2 \to [0, \infty)^*$ to $\ker \mu \times \ker \mu$, leaves a symmetric which yields the Scott topology. Now we claim that $X = \ker\mu$ is in fact a Hausdorff space: Let $m_n \in X$ be a sequence with two limits a and b. Now let $z \ll a$. Then since $\uparrow\!\!z \cap X$ is a Scott open set around a, $(\exists K_1)\ i \geq K_1 \Rightarrow m_i \in \uparrow\!\!z$. Since μ is a Lebesgue measurement and $\{m_i : i \geq K_1\} \cup \{a\}$ is a compact subset of the kernel,

$$(\exists\, \lambda > 0)\ \mu_\lambda(m_i) \subseteq \uparrow\!\!z$$

for all $i \geq K_1$. Since $b \in \ker \mu$, we can use the continuity of μ to choose an approximation $b_n \ll b$ with $\mu b_n < \lambda$. Now $\uparrow\!\!b_n \cap X$ is an open set around b and $m_n \to b$ so $(\exists K_2)\ j \geq K_2 \Rightarrow m_j \in \uparrow\!\!b_n \cap X$. Thus, if $n \geq \max\{K_1, K_2\}$,

$$b_n \ll m_n \ \&\ \mu_\lambda(m_n) \subseteq \uparrow\!\!z.$$

Then since $\mu b_n < \lambda$, we have $z \ll b_n \ll b$. Since z was an arbitrary approximation of a, we must have $a \sqsubseteq b$. However, since members of the kernel are maximal, $a = b$. Then any sequence in X with a limit has a unique limit. Since X is first countable, this implies that X is Hausdorff [8]. Finally, let $x \in X$, and suppose that (x_n) and (y_n) are sequences in X with $d(x, x_n) \to 0$ and $d(x_n, y_n) \to 0$. We need to show that $d(y_n, x) \to 0$. First, we know $x_n \to x$ since $d(x_n, x) \to 0$. Now let $U \subseteq P$ be a Scott open set around x. Then $(\exists K_1) i \geq K_1 \Rightarrow x_i \in U$. Since μ is a Lebesgue measurement and $\{x_i : i \geq K_1\} \cup \{x\}$ is a compact subset of the kernel,

$$(\exists \lambda > 0)\ \mu_\lambda(x_i) \subseteq U$$

for $i \geq K_1$. We know that $d(x_n, y_n) \to 0$, so

$$(\exists K_2)\ n \geq K_2 \Rightarrow d(x_n, y_n) < \lambda.$$

Now let $m \geq \max\{K_1, K_2\}$. By continuity of d, $(\exists z)\ z \ll x_m, y_m \ \&\ \mu z < \lambda$. But by our choice of m, we also know that $\mu_\lambda(x_m) \subseteq U$. Thus, $z \in U$, and since U is an upper set, $y_m \in U$. This proves that $y_n \to x$ and so $d(y_n, x) \to 0$. Now we may apply the metrization theorem for symmetrizable spaces to conclude that $\ker \mu$ is metrizable. $\Box$

The metrizability in the result above can be proven by constructing a metric, but it is a bit involved (see p.137 of [9]). The next result gives us an easy way to spot Lebesgue measurements in practice.

Theorem 5.4.2 *Any measurement $\mu : P \to [0,\infty)^*$ with the property that for all consistent pairs $x, y \in P$,*

$$(\,\exists\, z \sqsubseteq x, y\,)\; \mu z \leq 2 \cdot \max\{\mu x, \mu y\}$$

is a Lebesgue measurement. Furthermore, there is a metric ρ on $\ker \mu$ yielding the Scott topology such that

$$d(\mu)(x,y)/4 \leq \rho(x,y) \leq d(\mu)(x,y)$$

for all $x, y \in \ker \mu$.

proof Once again we are assuming without loss of generality that $d = d(\mu)$ is definable on P. Suppose that

$$d(x,z) < \varepsilon \;\&\; d(z,y) < \varepsilon.$$

Then $(\exists a \ll x, z)\mu a < \varepsilon$ and $(\exists b \ll z, y)\mu b < \varepsilon$. Since a and b are consistent, $(\exists c \sqsubseteq a, b)\mu c \leq 2{\cdot}\max\{\mu a, \mu b\} < 2\varepsilon$. Then $c \ll x, y$ so $d(x,y) \leq \mu c < 2\varepsilon$. By Frink's lemma, there is a function $\rho : P^2 \to [0,\infty)$, which inherits symmetry from d, satisfies the triangle inequality and for which

$$\frac{d(x,y)}{4} \leq \rho(x,y) \leq d(x,y)$$

for all $x, y \in \ker \mu$. Then $\rho(x,y) = 0 \Leftrightarrow x = y \in \ker \mu$, which proves that ρ is a metric. The inequality also makes it easy to see that ρ yields the Scott topology since d does. Now let $K \subseteq \ker \mu$ be Scott compact and let $U \subseteq P$ be a Scott open set containing K. First, for all $x \in K$, there is $\varepsilon_x > 0$ with $x \in B_{\varepsilon_x}(x) \subseteq U$, since $\mu \to \sigma_{\ker \mu}$ and $B_{\varepsilon_x}(x) = \uparrow\!\mu_{\varepsilon_x}(x)$. Then

$$\{B_{\frac{\varepsilon_x}{8}}(x) : x \in K\}$$

is an open cover of K. Taking a finite subcover,

$$\{B_{\frac{\varepsilon_i}{8}}(x_i) : x_i \in K, 1 \leq i \leq n\},$$

we choose $0 < \lambda < \min\{\frac{\varepsilon_i}{8}\}$. Now let $y \in B_\lambda(x)$ where $x \in K$ is arbitrary. Then $d(x, y) < \lambda$. Since $x \in K$, $(\exists x_i \in K)\, d(x, x_i) < \frac{\varepsilon_i}{8}$. Thus

$$\begin{aligned} \frac{d(y, x_i)}{4} \leq \rho(y, x_i) &\leq \rho(y, x) + \rho(x, x_i) \\ &\leq d(y, x) + d(x, x_i) \\ &< \lambda + \frac{\varepsilon_i}{8} < 2(\frac{\varepsilon_i}{8}) \end{aligned}$$

which shows $y \in B_{\varepsilon_i}(x_i) \subseteq U$. Then $\mu_\lambda(x) \subseteq \uparrow\mu_\lambda(x) = B_\lambda(x) \subseteq U$. This proves that μ is a Lebesgue measurement. $\square$

The last result is valuable not only because it guarantees that μ is a Lebesgue measurement, but because it also gives us a Lipschitz isomorphism between $d(\mu)$ and the Frink metric ρ: This enables us to treat $d(\mu)$ as though it were a metric.

Corollary 5.4.1 *Any measurement $\mu : P \to [0, \infty)^*$ with the property that for all consistent pairs $x, y \in P$,*

$$(\ \exists\, z \sqsubseteq x, y\ )\ \mu z \leq \mu x + \mu y$$

is a Lebesgue measurement. In fact, in this case, the symmetric $d(\mu)$ is actually a metric.

proof Let $x, y, z \in \ker \mu$ and $\varepsilon > 0$. First, choose $b \ll x, z$ and $c \ll z, y$ such that

$$d(x, z) \leq \mu b < d(x, z) + \frac{\varepsilon}{2}\ \&\ d(z, y) \leq \mu c < d(z, y) + \frac{\varepsilon}{2}.$$

Then b and c are consistent, $(\exists a \sqsubseteq b, c)\ \mu a \leq \mu b + \mu c$. Since $a \ll x, y$,

$$\begin{aligned} d(x, y) \leq \mu a &\leq \mu b + \mu c \\ &< (d(x, z) + \frac{\varepsilon}{2}) + (d(z, y) + \frac{\varepsilon}{2}) \\ &= d(x, z) + d(z, y) + \varepsilon \end{aligned}$$

But $\varepsilon > 0$ is arbitrary, so $d(x, y) \leq d(x, z) + d(z, y)$, which proves that d is a metric. That μ is a Lebesgue measurement follows from the last theorem since $\mu x + \mu y \leq 2 \cdot \max\{\mu x, \mu y\}$ $\square$

The previous corollary addresses the situation which occurs most often, especially on Lawson compact domains.

Example 5.4.1 The interval domain $(\mathbf{I}\mathbb{R}, \mu)$, the upper space $(\mathbf{U}X, \text{diam})$ of a locally compact metric space and the Cantor set model $(\Sigma^\infty, \frac{1}{2^{|\cdot|}})$ are examples where the symmetric induced by the measurement is actually a metric. The same is true of $([S], \text{len})$ and $([\mathbb{N} \to \mathbb{N}], |\text{dom}|)$.

Finally, we obtain a domain theoretic metrization theorem.

Theorem 5.4.3 (The metrization theorem) *For a topological space X, the following are equivalent:*

(i) *X is metrizable.*

(ii) *X is the kernel of a Lebesgue measurement on a continuous poset.*

proof (ii) $\Rightarrow$ (i) has already been proven. We show that (i) $\Rightarrow$ (ii). Given a metric space (X, d), the formal ball model [7]

$$\mathbf{B}X = X \times [0, \infty)$$

ordered by

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leq r - s$$

is a continuous poset whose approximation relation is $(x, r) \ll (y, s) \Leftrightarrow d(x, y) < r - s$. The natural map

$$\pi : \mathbf{B}X \to [0, \infty)^*$$

$$\pi(x, r) = r$$

is a Lebesgue measurement which induces the Scott topology everywhere and has $\ker\pi = \max\mathbf{B}\text{X} \simeq X$. In Example 2.3.2, we saw that π is a measurement with $\pi \to \sigma_{\mathbf{B}X}$. It is shown in [7] that $\ker\pi = \max\mathbf{B}\text{X} \simeq \text{X}$. Finally, to see that π is a Lebesgue measurement, let $(x, r), (y, s) \in \mathbf{B}X$ be consistent. Then $\exists\, (z, 0) \in \mathbf{B}X$ with $(x, r), (y, s) \sqsubseteq (z, 0)$. Setting $t = 2 \cdot \max\{r, s\}$, we see that

$$(z, t) \sqsubseteq (x, r), (y, s) \;\&\; \pi(z, t) \leq 2 \cdot \max\{\pi(x, r), \pi(y, s)\}.$$

By our last theorem, π is a Lebesgue measurement. $\square$

5.5 Complete Measurements

We have seen that the kernel of a Lebesgue measurement on a continuous poset is metrizable. Moreover, since every metric space can be represented in this fashion, we know that the kernel has no other topological properties. In particular, it need not be completely metrizable. In this section we will distinguish the completely metrizable spaces from the metrizable ones while remaining in the setting of continuous posets with measurements. However, we will notice that in doing so, the concept of a domain seems to be unavoidable.

Lemma 5.5.1 *A sequence (x_n) in the kernel of a Lebesgue measurement μ on a continuous poset P converges to $x \in \ker \mu$ if and only if*

(i) *(x_n) is Cauchy with respect to $d(\mu)$, and*

(ii) *There is a subsequence (x_{n_k}) and an increasing sequence (a_k) in P with $a_k \sqsubseteq x_{n_k}$, $\mu a_k \to 0$ and $\bigsqcup a_k = x$.*

proof Suppose that $x_n \to x$. First we show that (x_n) is Cauchy with respect to $d = d(\mu)$. To this end, let $\varepsilon > 0$. Since $x \in \ker \mu$, there is an increasing sequence (a_n) with $a_n \ll x$ and $\bigsqcup a_n = x$. By continuity of μ, choose K with $\mu a_K < \varepsilon$. Now $a_K \ll x$ and $x_n \to x$, so $(\exists\, N)\; i \geq N \Rightarrow x_i \in \Uparrow\!a_K$. Thus, if $i, j \geq N$, then

$$d(x_i, x_j) \leq \mu a_K < \varepsilon.$$

This proves that (x_n) is Cauchy. For (ii), choose $x_{n_k} \in \Uparrow\!a_k$ for all k. In the other direction, assume that (i) and (ii) hold. It is easy to see that $x_{n_k} \to x$. Now let U be a Scott open set around x. Then $(\exists K_1)\, n_i \geq K_1 \Rightarrow x_{n_i} \in U$. The set $\{x_{n_i} : n_i \geq K_1\} \cup \{x\}$ is compact and μ is a Lebesgue measurement so

$$(\exists \lambda > 0)\; B_\lambda(x_{n_i}) \subseteq U,$$

for $n_i \geq K_1$. Since (x_n) is Cauchy with respect to d,

$$(\exists K_2)\; m, n \geq K_2 \Rightarrow d(x_m, x_n) < \lambda.$$

Now let $j \geq \max(K_1, K_2)$. (x_{n_k}) is a subsequence, so $n_{k+1} > n_k$ for all k, which means we can choose a least $n_i > j$. Then $n_i, j \geq K_2$ implies $d(x_{n_i}, x_j) < \lambda$. But now $n_i > K_1$, which gives $x_j \in B_\lambda(x_{n_i}) \subseteq U$. Thus, $x_n \to x$. $\square$

Thus, for a Lebesgue measurement μ, $d(\mu)$ is complete iff Cauchy sequences along the top may be replaced with increasing sequences in the poset. This is exactly what ought to be true of the *natural metric* at the top.

Lemma 5.5.2 *Let* $\mu : P \to [0,\infty)^*$ *be a measurement on a continuous poset* P *whose induced symmetric* $d(\mu)$ *is defined. For a sequence* (x_n) *in* $\ker\mu$*, the following are equivalent:*

(i) (x_n) *is Cauchy with respect to* $d(\mu)$*.*

(ii) $(\forall\,\varepsilon > 0\,)(\,\exists\,K \in \mathbb{N}\,)(\,i, j \geq K \Rightarrow (\exists\,a \ll x_i, x_j)\,\mu a < \varepsilon\,)$.

Definition 5.5.1 Let $\mu : P \to [0,\infty)^*$ be a measurement on a continuous poset. A sequence (x_n) in $\ker\mu$ is *Cauchy* provided that for all $\varepsilon > 0$,

$$(\,\exists\,K \in \mathbb{N}\,)(\,i, j \geq K \Rightarrow (\exists\,a \ll x_i, x_j)\,\mu a < \varepsilon\,).$$

The measurement μ is *complete* if every Cauchy sequence in $\ker\mu$ has a limit in $\ker\mu$.

The definition given above has the advantage that it does not require $d(\mu)$ to exist. Of course, if each pair of elements in P is bounded from below, then $d(\mu)$ is defined, so μ is complete iff $d(\mu)$ is complete. In general, μ is complete on P iff $\mu_\perp$ is complete on $P_\perp$.

Example 5.5.1 Consider the upper space of the open unit interval $\mathbf{U}(0,1)$, which becomes a countably based Scott domain if we add a $\perp$, and as a measurement take

$$\mu : \mathbf{U}(0,1) \to [0,\infty)^*$$

$$\mu K = \text{diam}\, K$$

which we derive from the Euclidean metric on $(0,1)$. The natural symmetric is given by

$$d(\mu)(\{a\},\{b\}) = |b-a|$$

which of course is not complete. However, the space $\ker\mu \simeq (0,1)$ is locally compact and metrizable, so it does admit a complete metric. (If we choose such a complete metric for $(0,1)$, and use it to derive μ, then $d(\mu)$ will be complete.)

From the last example we see that measurements have the potential to behave at least as irrationally as metrics. What makes this situation worse than the metric case is that there are no metrics around! Even for a Lebesgue measurement μ, we have no guarantee that $d(\mu)$ is a metric. Consequently, it is not at all clear that the completeness of a measurement has anything to do with the complete metrizability of its kernel.

Theorem 5.5.1 *Let $\mu : P \rightarrow [0,\infty)^*$ be a measurement whose kernel is metrizable. Then the following are equivalent:*

(i) $\ker \mu$ *is completely metrizable.*

(ii) *There is a complete measurement $\lambda : P \rightarrow [0,\infty)^*$ with* $\ker \lambda = \ker \mu$.

proof (i) $\Rightarrow$ (ii) Let d be a bounded complete metric on $X = \ker \mu$. For each $n \geq 1$, we set

$$U_n = \bigcup \{\Uparrow x : \mathrm{diam}(\Uparrow x \cap X) < \frac{1}{2^n}\}.$$

It is clear that (U_n) is a decreasing sequence of Scott open sets in P. Also observe that $X \subseteq \bigcap U_n$: For if $x \in X$, then $x \in B_\varepsilon(x)$ for any $\varepsilon > 0$, so for each $n \geq 1$,

$$(\exists\, b_n \ll x)\, x \in \Uparrow b_n \cap X \subseteq B_{\frac{1}{2^{n+1}}}(x).$$

Then $x \in \Uparrow b_n \subseteq U_n$ for all $n \geq 1$. Next, there is a Scott continuous map $\sigma : P \rightarrow [0,\infty)^*$ with $\ker \sigma = \bigcap U_n$ satisfying

$$\sigma x \leq \frac{1}{2^n} \Leftrightarrow x \in U_n,$$

for $n \geq 1$. Then the mapping

$$\lambda : P \rightarrow [0,\infty)^*$$

$$x \mapsto \mu x + \sigma x$$

is Scott continuous with kernel $\ker \lambda = \ker \mu \cap \ker \sigma = X$. Furthermore, $\mu x \leq \lambda x$ for all x, so for any $x \in X$ and $\varepsilon > 0$, $x \in \lambda_\varepsilon(x) \subseteq \mu_\varepsilon(x)$. But μ is a measurement so this implies that λ is too. We claim that λ is complete. To this end, let (x_n) be a sequence in $X = \ker \lambda$ which is Cauchy with respect to λ. Then for any $n \geq 1$,

$$(\exists\, K_n)(\, i,j \geq K_n \Rightarrow (\exists\, a \ll x_i, x_j)\, \lambda a < \frac{1}{2^n}\,).$$

Now fix integers $i, j \geq K_n$. Then there exists an element $a \ll x_i, x_j$ with $\lambda a < \frac{1}{2^n}$. Since $\sigma a \leq \lambda a < \frac{1}{2^n}$, we know that $a \in U_n$. By the definition of U_n, $(\exists\, b \ll a)$ such that

$$\mathrm{diam}(\uparrow\!\!\!\uparrow b \cap X) < \frac{1}{2^n}.$$

But $b \ll a \ll x_i, x_j$ so $d(x_i, x_j) \leq \mathrm{diam}(\uparrow\!\!\!\uparrow b \cap X) < \frac{1}{2^n}$. Then (x_n) is Cauchy with respect to the complete metric d on X, so it has a limit in X. Consequently, every sequence Cauchy with respect to λ converges.

(ii) $\Rightarrow$ (i): Let μ be a complete measurement. In the usual way, we may assume that $d(\mu)$ exists and is therefore complete. Finally, write $X = \ker \mu$. We are going to use Engelking's characterization of Čech-completeness to prove that X is Čech-complete assuming that it is Tychonoff. First recall that given $a \in X$ and $\varepsilon > 0$,

$$(\exists \delta > 0)(\forall x, y \in X)\; d(x, a) < \delta \;\&\; d(y, a) < \delta \Rightarrow d(x, y) < \varepsilon.$$

If not, one can find two sequences (x_n), (y_n) and $\varepsilon > 0$ such that $x_n \to a$, $y_n \to a$ and $d(x_n, y_n) \geq \varepsilon$ for all n. Thus, for all $a \in X$, there is a $\delta_n(a) > 0$ such that for all $x, y \in X$,

$$d(x, a) < \delta_n(a) \;\&\; d(y, a) < \delta_n(a) \Rightarrow d(x, y) < \frac{1}{2^n}$$

This gives us a countable family of open covers

$$\mathcal{R}_n = \{B_{\delta_n(x)}(x) \cap X : x \in X\}.$$

Suppose $\mathcal{F}$ is a collection of closed sets of X with the finite intersection property such that

$$(\forall i)(\exists C_i \in \mathcal{F})(\exists U_i \in \mathcal{R}_i)(C_i \subseteq U_i)$$

We need to show that $\bigcap \mathcal{F} \neq \emptyset$. Let $F_n = \bigcap_{i=1}^n C_i$ for all $n \geq 1$. Then (F_n) is a decreasing sequence of nonempty closed sets and

$$F_n \subseteq U_n = B_{\delta_n(a_n)}(a_n).$$

Choose a point $x_n \in F_n$. We claim that (x_n) is Cauchy:

$$\begin{aligned} i, j \geq n \;&\Rightarrow\; x_i, x_j \in F_n \\ &\Rightarrow\; x_i, x_j \in B_{\delta_n(a_n)}(a_n) \\ &\Rightarrow\; d(x_i, a_n) < \delta_n(a_n) \;\&\; d(x_j, a_n) < \delta_n(a_n) \\ &\Rightarrow\; d(x_i, x_j) < \frac{1}{2^n} \end{aligned}$$

Then (x_n) converges to some $x \in X$ since $d(\mu)$ is complete. The sets (F_n) are closed and all contain a tail of (x_n) so $x \in \bigcap F_n$. If $y \in \bigcap F_n$, then

$$(\forall n)\, d(x, a_n) < \delta_n(a_n) \ \&\ d(y, a_n) < \delta_n(a_n)$$

so $d(x, y) < \frac{1}{2^n}$. This proves that $\bigcap F_n = \{x\}$. Now let $F \in \mathcal{F}$ be arbitrary. By the finite intersection property of $\mathcal{F}$, $F \cap F_n \neq \emptyset$. Then since $\{F \cap F_n\}$ is a decreasing sequence of nonempty closed sets with $F \cap F_n \subseteq U_n$, the same reasoning above shows that

$$\emptyset \neq \bigcap (F \cap F_n) = F \cap \bigcap F_n = F \cap \{x\}.$$

Thus $x \in F$ and so $\bigcap \mathcal{F} \neq \emptyset$. Since X is metrizable, it is Tychonoff, and so X is Čech-complete. But a Čech-complete metric space is completely metrizable. $\Box$

Observe an immediate corollary of the last result: For *any* complete metric d on $\ker\mu$, one can find a measurement λ which has the same Cauchy sequences as d and the same kernel as μ. (For the unproven direction, any Cauchy sequence with respect to d converges and hence must be Cauchy with respect to λ.) The next corollary implies that λ also inherits the Lebesgue property from μ.

Corollary 5.5.1 *If $\mu : P \to [0, \infty)^*$ is a measurement whose kernel is completely metrizable, then there is a complete measurement $\lambda : P \to [0, \infty)^*$ with $\ker \lambda = \ker \mu$ and $\mu x \leq \lambda x$ for all $x \in P$.*

Corollary 5.5.2 *If $\mu : P \to [0, \infty)^*$ is a complete measurement on a continuous poset, then $\ker \mu$ is Tychonoff iff it is Čech-complete.*

Proposition 5.5.1 *For a topological space X, the following are equivalent:*

(i) *X is completely metrizable.*

(ii) *X is the kernel of a complete Lebesgue measurement on a continuous poset.*

(iii) *X is the kernel of a complete Lebesgue measurement on a continuous dcpo.*

proof (iii) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (i): As the kernel of a Lebesgue measurement, X is metrizable. By the last result, any metric space which is the kernel of a complete measurement is completely metrizable. (i) $\Rightarrow$ (iii): Let (X, d) be a metric space with complete metric d. The formal ball model $\mathbf{B}X$ is now a continuous dcpo with Lebesgue measurement $\pi(x, r) = r$. For a point $x \in X$ we will write $\bar{x} = (x, 0) \in \ker \pi$. We claim the induced symmetric $d(\pi)$ is complete. To see this, let $x, y \in X$ and $(z, r) \ll \bar{x}, \bar{y}$. Then $d(x, y) \leq 2 \cdot r = 2 \cdot \pi(z, r)$ so

$$\frac{d(x,y)}{2} \leq \inf\{\pi(z,r) : (z,r) \ll \bar{x}, \bar{y}\} = d(\pi)(\bar{x}, \bar{y})$$

In fact, since $(x, d(x,y) + \frac{1}{n}) \ll \bar{x}, \bar{y}$, we easily derive

$$d(\pi)(\bar{x}, \bar{y}) \leq d(x, y).$$

This provides a Lipschitz isomorphism between (X, d) and $(\max \mathbf{B}\mathrm{X}, d(\pi))$. Since d and $d(\pi)$ have the same Cauchy sequences and d is complete, $d(\pi)$ is complete. $\square$

Corollary 5.5.3 *A space is Polish iff it is the kernel of a complete Lebesgue measurement on an ω-continuous dcpo.*

It is interesting to note that the completeness of μ is actually a kind of sobriety in disguise.

Proposition 5.5.2 *Let $\mu : P \to [0, \infty)^*$ be a measurement on a continuous poset whose relative Scott and Lawson topologies agree on* $\ker \mu$. *If μ is complete, then every increasing sequence (x_n) in $\downarrow \ker \mu$ with $\mu x_n \to 0$ has a supremum.*

proof Let (x_n) be increasing in $\downarrow \ker \mu$ with $\mu x_n \to 0$. Then $\exists\, a_n \in \ker \mu$ with $x_n \sqsubseteq a_n$. The sequence (a_n) is Cauchy with respect to μ so it has a limit $x \in \ker \mu$. However, $\uparrow x_n \cap \ker \mu$ is a Scott closed set for each n, so $x_n \sqsubseteq x$. Then since x is an upper bound for the (x_n) and $\lim \mu x_n = \mu x = 0$, basic properties of measurement give that $\bigsqcup x_n = x$. $\square$

Taking the rounded ideal completion of P in the result above now shows that $\ker \mu$ is actually a G_δ in a domain. But notice that we had to assume two things to achieve this result: First, that the relative Scott and Lawson topologies on $\ker \mu$ agree, and second, that the measurement μ is complete.

The first assumption implies $\ker \mu$ is Tychonoff, and as we have seen, the completeness of the measurement now implies that $\ker \mu$ must be Čech-complete. In a similar way, a metric space X is completely metrizable iff $\mathbf{B}X$ is a domain. These two examples, though they both assume the Scott and Lawson topologies agree at the top, suggest the following question: Is a metric space completely metrizable *if and only if* it is a G_δ subset of a continuous dcpo? We will return to this question at the end of Chapter 6.

5.6 Existence of Complete Lebesgue Measurements

In the last section, we gave a characterization of the completely metrizable spaces. In particular, we saw that a space is Polish iff it is the kernel of a complete Lebesgue measurement on an ω-continuous dcpo. Jimmie Lawson [16] has proven that the Polish spaces are precisely the spaces at the top of countably based domains *whose relative Scott and Lawson topologies agree at the top*:

$$(\,\forall\, x \in D\,)\uparrow\!x \cap \max D \text{ is a Scott closed subset of } \max D.$$

The formal ball model $\mathbf{B}X$ of a Polish space X provides one important example of this, while the countably based Scott domains yield another. The point of this section is to study the relationship between these two classes of domains.

Theorem 5.6.1 *Every ω-continuous dcpo D whose relative Scott and Lawson topologies agree at the top admits a Lebesgue measurement*

$$\mu : D \to [0, \infty)^*$$

with $\ker \mu = \max D$ *and* $\mu \to \sigma_D$.

proof The reader is advised to see [16] for a nice illustration of important techniques. The ω-continuous dcpo D embeds as the set of coprimes in the continuous lattice of its Scott closed sets $\Gamma(D)$. The Lawson closure of its image in $\Gamma(D)$ is an order compactification denoted by $F(D)$ — this is called the *Fell order compactification.* Since D is ω-continuous, $F(D)$ is a compact metrizable partially ordered space, and so it admits a metric $\bar{d}$ which is radially convex:

$$x \sqsubseteq y \sqsubseteq z \Rightarrow \bar{d}(x, z) = \bar{d}(x, y) + \bar{d}(y, z)$$

for $x, y, z \in F(D)$. The restriction of $\bar{d}$ to D we denote by d. Specifically,

$$d(x, y) = \bar{d}(\downarrow x, \downarrow y)$$

for $x, y \in D$. This metric is also radially convex and yields the Lawson topology on D. Now define $\mu : D \to [0, \infty)^*$ by

$$\mu x = \sup\{\ \bar{d}(\downarrow x, y) : x \in y \in F(D)\ \}$$

Note that the supremum exists since the space $F(D)$ is compact. In Chapter 2, we proved that $\mu \to \sigma_D$ and $\ker \mu = \max D$. We need to show that μ is a Lebesgue measurement, so let $K \subseteq \max D$ be Scott compact and $U \subseteq D$ be a Scott open set containing K. The relative Scott and Lawson topologies on $\max D$ agree, so K is Lawson compact in D. Since U is Lawson open and the metric d yields the Lawson topology, the Lebesgue covering lemma from topology implies that

$$(\exists \lambda > 0)(\forall x \in K)\ B_\lambda(x) \subseteq U.$$

We claim that $\mu_\lambda(x) \subseteq U$ for all $x \in K$. If $y \in \mu_\lambda(x)$, then since $y \sqsubseteq x$,

$$d(x, y) \leq \mu y - \mu x = \mu y - 0 = \mu y < \lambda$$

which means $y \in B_\lambda(x) \subseteq U$. Thus, $\mu_\lambda(x) \subseteq U$ for all $x \in K$, which proves that μ is a Lebesgue measurement. □

We have seen that the kernel of a measurement is a G_δ subset of the top. We even characterized G_δ subsets as the kernels of Scott continuous mappings on domains. In view of this characterization, it is natural to wonder how close G_δ's are to various forms of measurement.

Proposition 5.6.1 *Let D be an ω-continuous dcpo whose relative Scott and Lawson topologies agree on the top. For a subset $X \subseteq \max D$, the following are equivalent:*

(i) *X is a G_δ subset of* $\max D$.

(ii) *X is a G_δ subset of D.*

(iii) *X is the kernel of a measurement on D.*

(iv) *X is the kernel of a Lebesgue measurement on D.*

(v) *X is the kernel of a complete Lebesgue measurement on D.*

proof (i) $\Rightarrow$ (v): By the existence theorem, let $\mu : D \to [0,\infty)^*$ be a Lebesgue measurement with $\ker\mu = \max D$. Since X is a G_δ in $\max D$ and $\max D$ is a G_δ in D, X is a G_δ in D. Adding the mapping representing X with μ yields a measurement $\lambda : D \to [0,\infty)^*$ with $\ker\lambda = X \cap \ker\mu = X$ and $\mu x \leq \lambda x$ for all $x \in D$. The inequality implies that λ is a Lebesgue measurement since μ is. By Lawson's theorem, $\max D$ is Polish. Hence X is Polish as as G_δ subset of a Polish space. Since $\ker\lambda = X$ is completely metrizable, there is a complete measurement $\pi : D \to [0,\infty)^*$ with $\ker\pi = \ker\lambda$ and $\lambda x \leq \pi x$ for all $x \in D$. Once again, the inequality implies that π is a Lebesgue measurement since λ is, which shows that π is the desired complete Lebesgue measurement. $\Box$

In 5.2 we saw a domain whose maximal elements are the kernel of a (complete) measurement but are not the kernel of a Lebesgue measurement. It is not known if (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) hold in general or not.

Corollary 5.6.1 *Every ω-continuous dcpo D whose relative Scott and Lawson topologies agree at the top admits a complete Lebesgue measurement $\mu : D \to [0,\infty)^*$ with* $\ker\mu = \max D$.

proof As an open subset of itself, $\max D$ is a G_δ subset of $\max D$. $\Box$

Consequently, the class of continuous posets with (complete) Lebesgue measurements naturally includes all countably based domains where the Scott and Lawson topologies agree at the top.

5.7 Models of Spaces

Studying the topological structure of the kernel may be regarded as one instance of a more general problem: Which spaces may be represented as the maximal elements of a continuous dcpo? We say that such spaces have *domain theoretic models.*

Definition 5.7.1 A *model* of a topological space X is a continuous dcpo D and a homeomorphism

$$\phi : X \to \max D$$

where $\max D$ carries its inherited Scott topology from D.

The objective of a model is to isolate the space within a domain in a useful way. Typically the space is distinguished as being the *entire* set of maximal elements in a domain. However, in studying the kernel of a measurement, we learned that a space can also be isolated as a G_δ *subset* of the top. In this section we want to point out certain results that are known about G_δ sets in general. At the same time, it will give the reader some feel for what life without measurements is like.

Proposition 5.7.1 *Let $X \subseteq \max D$ be a G_δ in a continuous dcpo. Then X is a first countable T_1 space. If $X = \max D$, X is a Baire space.*

proof First, by Corollary 2.4.1, we know that there exists a Scott continuous map $\mu : D \to [0,\infty)^*$ with $\ker\mu = X$. For $x \in X$, we can use the directedness of $\twoheaddownarrow x$ combined with the continuity of μ to choose an increasing sequence (x_n) such that $(\forall n)(\, x_n \ll x \;\&\; \mu x_n < \frac{1}{n}\,)$. Then $\mu(\bigsqcup x_n) = 0$ by continuity of μ. Since $\bigsqcup x_n \in X \subseteq \max D$ and $\bigsqcup x_n \sqsubseteq x$, $\bigsqcup x_n = x$. This proves that $\{\twoheaduparrow x_n : n \in \mathbb{N}\}$ is a countable basis at x. Then D is first countable at x and so this property is inherited by the subspace X.

For the other remark, a continuous dcpo D in its Scott topology is a Baire space [10]. If $X = \max D$, then X is a dense G_δ in a Baire space and hence a Baire space in its own right. $\Box$

Paracompactness is that topological idea which allows us to extend the local to the global within a topological space. For example, a locally metrizable paracompact space is metrizable (think of a manifold). Our two favorite examples of paracompacta are metric spaces and compact Hausdorff spaces.

Theorem 5.7.1 *Let D be a Scott domain and $X \subseteq \max D$ be a G_δ in D. Then the following are equivalent:*

(i) *X is paracompact.*

(ii) *X is metrizable.*

(iii) *X is completely metrizable.*

proof (i) $\Rightarrow$ (iii) A Čech-complete paracompact space is metrizable iff it has a G_δ diagonal (see [8], exercise 5.1.I). Since X is a G_δ in D, $\exists \mu : D \to [0,\infty)^*$ which is Scott continuous and has $\ker\mu = X$. Since D is a Scott domain, it has a bottom element, so the mapping

$$d : D^2 \to [0,\infty)^*$$

$$d(x, y) = \inf\{\mu z : z \ll x, y\}$$

is well-defined. The monotonicity of μ implies d is Scott continuous. Since D is a Scott domain, $d^{-1}(0) = \{(x, x) : x \in X\}$ is a G_δ in D^2 and hence in X^2. The Lawson topology on D is compact, so X is a G_δ in a compact Hausdorff space, which makes it Čech-complete. Then X is a Čech-complete paracompact space with a G_δ diagonal, so it is metrizable. But a Čech-complete metric space is completely metrizable. □

First, the nicest of all paracompact spaces may sit at the top of a Scott domain without being a G_δ.

Example 5.7.1 Let $X = [0, 1]^2$ with the dictionary order. Then X is a first countable, hereditarily separable, compact Hausdorff space which is not metrizable. Its upper space $\mathbf{U}X$ is a Scott domain whose maximal elements

$$X \simeq \max \mathbf{U}X$$

cannot form a G_δ set with respect to the Scott topology: By the previous result, if $\max \mathbf{U}X$ were a G_δ in $\mathbf{U}X$, X would have to be metrizable.

The last example can be summarized as follows.

Corollary 5.7.1 *For a compact Hausdorff space X, the following are equivalent:*

(i) *X is metrizable.*

(ii) *X is second countable.*

(iii) *$\mathbf{U}X$ is an ω-continuous Scott domain.*

(iv) *There is a measurement $\mu : \mathbf{U}X \to [0, \infty)^*$ with $\ker \mu = \max \mathbf{U}X$.*

(v) *$\max \mathbf{U}X$ is a G_δ in $\mathbf{U}X$.*

On the other hand, being a G_δ in a Scott domain does not even imply the space is normal, let alone paracompact!

Example 5.7.2 Let X be the Cantor set model Σ^∞ regarded as a space in its μ topology. The natural measurement on Σ^∞ is

$$\mu : \Sigma^\infty \to [0, \infty)^*$$

$$\mu s = \frac{1}{2^{|s|}}$$

which induces the Scott topology everywhere. By the invariance of measurement, we know that a basis for the μ topology is $\{\mu_\varepsilon(x) : x \in \Sigma^\infty, \varepsilon > 0\}$ where we recall that

$$\mu_\varepsilon(x) = \{y \in \Sigma^\infty : y \sqsubseteq x \;\&\; \mu y < \varepsilon\}.$$

Σ^∞ is an ω-continuous dcpo with $\max \Sigma^\infty$ uncountable so the results in Chapter 3 show that X is a zero dimensional, first countable, separable Hausdorff space which is *not normal.* In this special case, each set $\mu_\varepsilon(x)$ is a convergent sequence with its limit x adjoined and hence μ compact. Then X is locally compact and its upper space $\mathbf{U}X$ is an algebraic Scott domain without a countable base. We are going to construct a measurement λ on $\mathbf{U}X$ with $\ker \lambda = \max \mathbf{U}X \simeq X$. For each $n \geq 1$, set

$$U_n = \bigcup_{s \in \max \Sigma^\infty} \uparrow_{\mathbf{U}X} \mu_{\frac{1}{2^n}}(s).$$

Notice that $\mu_\varepsilon(x)$ is compact and open in X so $\mu_\varepsilon(x) \in K(\mathbf{U}X)$ and thus each U_n is a Scott open set in $\mathbf{U}X$. For $s \in \max \Sigma^\infty$, $\{s\} \subseteq \mu_{\frac{1}{2^n}}(s)$ for all n, so $\{s\} \in \bigcap U_n$. On the other hand, if $K \in \bigcap U_n$ then

$$(\forall n)(\exists s_n \in \max \Sigma^\infty)\; K \in \uparrow_{\mathbf{U}X} \mu_{\frac{1}{2^n}}(s_n).$$

Thus, $K \subseteq \mu_{\frac{1}{2^n}}(s_n)$ for all n. If $s \in K$, then $s \sqsubseteq s_n$ and $\mu s < \frac{1}{2^n}$ for all n, so $\mu s = 0$. Hence $s \in \max \Sigma^\infty$ and we must have $s = s_n$ for all n. Since $K \neq \emptyset$, we conclude $K = \{s\}$ for some $s \in \max \Sigma^\infty$. Consequently,

$$\bigcap U_n = \{\{s\} : s \in \max \Sigma^\infty\}.$$

Next, any finite string $a \in \Sigma^\infty$ yields a compact open subset $\{a\}$ of X, so $\{a\} \in K(\mathbf{U}X) \cap \max \mathbf{U}X$, which proves that

$$V = \{\{a\} : a \in \Sigma^\infty \textit{ finite}\}$$

is a Scott open set in $\mathbf{U}X$. Then

$$\begin{aligned}
\max \mathbf{U}X &= \{\{x\} : x \in X\} \\
&= \{\{a\} : a \in \Sigma^\infty \textit{ finite}\} \bigcup \{\{s\} : s \in \max \Sigma^\infty\} \\
&= \bigcap_{n \geq 1} (V \cup U_n)
\end{aligned}$$

is a G_δ subset of $\mathbf{U}X$. The sets $V_n = V \cup U_n$ form a decreasing sequence of Scott open sets so there is a Scott continuous map $\lambda : \mathbf{U}X \to [0,\infty)^*$ with $\ker \lambda = \max \mathbf{U}X$ satisfying $\lambda x \leq \frac{1}{2^n} \Leftrightarrow x \in V_n$, for all $n \geq 1$. We claim that this mapping is a measurement. To prove this, we only need to consider compact elements of $\mathbf{U}X$ since it is algebraic, so let $K \in K(\mathbf{U}X)$ with $K \ll \{s\}$ for some $s \in \Sigma^\infty$. If s is finite in Σ^∞, then

$$\begin{aligned} L \ll \{s\} \ \&\ \lambda L < \mu s &\Rightarrow s \in L \ \&\ \lambda s \leq \frac{1}{2^{|s|+1}} \\ &\Rightarrow s \in L \ \&\ L \in V_{|s|+1} \\ &\Rightarrow L = \{s\} \end{aligned}$$

so we can assume that $s \in \max \Sigma^\infty$. Using the fact that K is open in X, $(\exists\, \varepsilon > 0)\ s \in \mu_\varepsilon(s) \subseteq K$. Now choose the least $n \geq 1$ with $\frac{1}{2^n} < \varepsilon$ and observe

$$\begin{aligned} L \ll \{s\} \ \&\ \lambda L < \frac{1}{2^n} &\Rightarrow s \in L \ \&\ L \in V_n \\ &\Rightarrow (\exists s_n \in \max \Sigma^\infty)\ L \in \uparrow_{\mathbf{U}X} \mu_{\frac{1}{2^n}}(s_n) \\ &\Rightarrow L \subseteq \mu_{\frac{1}{2^n}}(s) \subseteq \mu_\varepsilon(s) \subseteq K \\ &\Rightarrow K \ll L \end{aligned}$$

This finishes the proof that λ is a measurement. Observe that X is not hereditarily separable even though it is separable. (By taking the upper space of an appropriate locally compact metric space, we see that the kernel of a measurement on a Scott domain need not be separable either.) Lastly, there are Lawson compact domains with measurements which do not possess Lebesgue measurements.

In the last example, the set of finite strings form a dense subset of X which is completely metrizable in its relative topology. It is interesting if not amazing that this always happens.

Theorem 5.7.2 *If D is a Scott domain and $X \subseteq \max D$ is a G_δ in D, then there exists $Y \subseteq X$ such that*

(i) *Y is dense in X.*

(ii) *Y is a G_δ in D.*

(iii) *Y is completely metrizable in its relative Scott topology.*

proof First, since X is Čech-complete it contains a dense Čech-complete paracompact subset Y (see [2], p.37). Since Y is Čech-complete and is densely embedded in the Tychonoff space X, Y is a G_δ in X. But X is a G_δ in D so Y is also a G_δ in D. By the last theorem, Y is completely metrizable as a paracompact G_δ in a Scott domain. □

Then G_δ subsets of Scott domains are spaces in which a completely metrizable space has been densely embedded. This idea will make more sense if we consider the countably based case.

Proposition 5.7.2 *For a topological space X, the following are equivalent:*

(i) *X is Polish.*

(ii) *There is an ω-continuous Scott domain D in which $X \subseteq \max D$ is a G_δ.*

(iii) *There is an ω-continuous Scott domain D in which $X \subseteq \max D$ is a dense G_δ.*

proof (iii) ⇒ (ii) is obvious. (ii) ⇒ (i): By (ii), X is embedded in a countably based Scott domain, so it is second countable. The Scott and Lawson topologies on $\max D$ agree so X inherits regularity from $\max D$. Then Urysohn's metrization theorem implies X is metrizable. But this means X is paracompact and every paracompact G_δ in a Scott domain is completely metrizable. Hence X is Polish. (i) ⇒ (iii): The space X being Polish embeds as a G_δ subset of the Hilbert cube

$$\mathbb{H} = \prod_{n \in \mathbb{N}} [0, 1/n]$$

which is a compact metric space as the countable product of compact unit intervals. Let Y be the closure of X in $\mathbb{H}$. Then X is a dense G_δ subset of the compact metric space Y. We note the following embeddings:

$$X \hookrightarrow Y \simeq \max \mathbf{U}Y \hookrightarrow \mathbf{U}Y.$$

Then X is a G_δ in Y and Y is a G_δ in $\mathbf{U}Y$ (by the last corollary), so by general topology X is a G_δ in $\mathbf{U}Y$. Next, X is dense in Y and Y is dense in $\mathbf{U}Y$ (as the set of maximal elements), so X is dense in $\mathbf{U}Y$. Since $\mathbf{U}Y$ is an ω-continuous Scott domain, we are done. □

Corollary 5.7.2 *A space X is Polish iff there is an ω-continuous Scott domain D and a Lebesgue measurement $\mu : D \to [0,\infty)^*$ with* $\ker \mu = \mathrm{X}$.

proof ($\Rightarrow$): The Polish space X embeds as a G_δ in the ω-continuous Scott domain $\mathbf{U}\mathbb{H}$. The diameter map diam : $\mathbf{U}\mathbb{H} \to [0,\infty)^*$ is a complete Lebesgue measurement and X is a G_δ in $\mathbf{U}\mathbb{H}$ so there exists a measurement $\mu : \mathbf{U}\mathbb{H} \to [0,\infty)^*$ with diam $\leq \mu$ and $\ker \mu = X$. The inequality proves that μ is Lebesgue. Alternately, one can simply appeal to the results of the previous section since the relative Scott and Lawson topologies on $\max D$ agree. $\Box$

Recall that a Čech-complete space is one which embeds as a G_δ subset of a compact Hausdorff space. We have seen that spaces may also be embedded as G_δ subsets of Scott domains (which have compact Lawson topology). We know that these two notions of embedding are not equivalent in general since there are Čech-complete spaces which are not first countable. It is somewhat remarkable then that they *are* equivalent for *second countable* Čech-complete spaces, i.e., Polish spaces.

5.8 Questions

(i) We have seen that the kernel of a measurement on a continuous poset is a Moore space assuming it is regular. Is every Moore space the kernel of a measurement on a continuous poset? (Hint: Try to make use of Nemytskij's characterization of Moore spaces (see [2], p.30)).

(ii) Can we prove that every countably based domain on which the relative Scott and Lawson topologies agree admits a Lebesgue measurement satisfying either one of the triangle inequality variants?

(iii) Given a measurement with the Frink triangle inequality, how do we construct a new one which satisfies the triangle inequality, and such that a Lipschitz isomorphism results between both versions of the space?

(iv) Can *every* complete metric space be embedded as a dense G_δ subset of a Scott domain?

(v) If $X \subseteq \max D$ is a G_δ subset of an arbitrary Scott domain, is there a measurement μ with $\ker \mu = X$?

(vi) Do the maximal elements of a countably based domain form a G_δ? (Hint: Every countably based domain admits $\mu : D \to [0, \infty)^*$ which is Scott continuous and induces the Scott topology everywhere. Hence, $x < y \Rightarrow \mu x > \mu y$. Now define

$$\lambda : D \to [0, \infty)^*$$

$$\lambda x = \sup\{\mu x - \mu y : x \sqsubseteq y\} = \mu x - \inf\{\mu y : y \in \uparrow x\}$$

This mapping is monotone and has $\ker \lambda = \max D$. When is it Scott continuous?)

(vii) The top of an ω-algebraic Scott domain is either $\mathbb{R} \setminus \mathbb{Q} \simeq \max [\mathbb{N} \to \mathbb{N}]$ or is locally compact at some point. Does a similar result hold in the non-algebraic case, or does this just happen because the spaces in question here are zero-dimensional and Polish?

(viii) If the maximal elements of an *arbitrary* Scott domain are completely metrizable, must they form a G_δ set in the domain?

(ix) Is there a Baire metric space which is not completely metrizable that has a G_δ model? Is there a model of the rationals? (Note that $\mathbb{Q}$ has no G_δ model.)

(x) We saw a measurement on an algebraic Scott domain whose kernel was not normal. Is there a measurement on a domain whose kernel is normal but not metrizable?

(xi) When is the Lawson topology on a continuous dcpo Čech-complete? (For countably based domains, the answer is *always*. What about domains with first countable Scott topology?)

Chapter 6

Measurements on the Convex Powerdomain

6.1 The Convex Powerdomain

A useful technique for constructing domains is to take the *ideal completion* of an *abstract basis.*

Definition 6.1.1 An *abstract basis* is given by a set B together with a transitive relation $<$ on B which is *interpolative,* that is,

$$M < x \Rightarrow (\,\exists\, y \in B\,)\; M < y < x$$

for all $x \in B$ and all finite subsets of B.

Abstract bases are covered in [1], which is where one also finds the following.

Definition 6.1.2 An *ideal* in $(B, <)$ is a nonempty subset I of B such that

(i) I is a lower set: $(\,\forall\, x \in B\,)(\,\forall\, y \in I\,)\; x < y \Rightarrow x \in I$.

(ii) I is directed: $(\,\forall\, x, y \in I\,)(\,\exists\, z \in I\,)\; x, y < z$.

The collection of ideals of an abstract basis $(B, <)$ ordered under inclusion is a partially ordered set called the *ideal completion* of B. We denote this poset by $\bar{B}$.

The set $\{y \in B : y < x\}$ for $x \in B$ is an ideal which leads to a natural mapping from B into $\overline{B}$, given by $i(x) = \{y \in B : y < x\}$.

Proposition 6.1.1 *If* $(B, <)$ *is an abstract basis, then*

(i) *Its ideal completion* $\bar{B}$ *is a dcpo.*

(ii) *For* $I, J \in \bar{B}$,

$$I \ll J \Leftrightarrow (\,\exists\, x, y \in B\,)\; x < y \;\&\; I \subseteq i(x) \subseteq i(y) \subseteq J.$$

(iii) $\bar{B}$ *is a continuous dcpo with basis* $i(B)$.

(iv) *If* $<$ *is reflexive, then* $\bar{B}$ *is algebraic.*

(v) *If* $<$ *is a partial order, then* $B \simeq K(\bar{B}) \simeq i(B)$.

If one takes any basis B of a domain D and restricts the approximation relation $\ll$ on D to B, they are left with an abstract basis $(B, \ll)$ whose ideal completion is D. Thus, all domains arise as the ideal completion of an abstract basis. We now use this technique to construct a domain called the *convex powerdomain.*

Definition 6.1.3 Let D be a continuous dcpo. For subsets $A, B \subseteq D$, we define relations

- $A \ll_{\mathrm{L}} B \Leftrightarrow (\forall a \in A)(\exists b \in B)\, a \ll b$
- $A \ll_{\mathrm{U}} B \Leftrightarrow (\forall b \in B)(\exists a \in A)\, a \ll b$
- $A \ll_{\mathrm{EM}} B \Leftrightarrow A \ll_{\mathrm{L}} B \;\&\; A \ll_{\mathrm{U}} B$

In the same way, we derive $\sqsubseteq_{\mathrm{L}}, \sqsubseteq_{\mathrm{U}}$ and $\sqsubseteq_{\mathrm{EM}}$ from the order $\sqsubseteq$ on D.

Definition 6.1.4 The nonempty finite subsets of a space X are denoted $\mathcal{P}_{\mathit{fin}}(X)$, while its nonempty compact subsets are written as $\mathcal{P}_{\mathit{com}}(X)$.

The set $\mathcal{P}_{\mathit{fin}}(D)$ together with $\ll_{\mathrm{EM}}$ is an abstract basis.

Definition 6.1.5 The *convex powerdomain* $\mathbf{C}D$ of a continuous dcpo D is the ideal completion of the abstract basis $(\mathcal{P}_{\mathit{fin}}(D), \ll_{\mathrm{EM}})$.

All topological statements about domains and their subspaces are made with respect to the Scott topology.

Definition 6.1.6 For a Scott compact $K \in \mathcal{P}_{com}(D)$, we set

$$K^* = \{F \in \mathcal{P}_{fin}(D) : F \ll_{\rm EM} K\}.$$

Notice that this operation is also defined for elements of $\mathcal{P}_{fin}(D)$.

Proposition 6.1.2 *For a continuous dcpo D, we have*

(i) *If $K \in \mathcal{P}_{com}(D)$, then $K^* = \{F \in \mathcal{P}_{fin}(D) : F \ll_{\rm EM} K\} \in \mathbf{C}D$.*

(ii) *For ideals $I, J \in \mathbf{C}D$,*

$$I \ll J \Leftrightarrow (\exists F, G \in \mathcal{P}_{fin}(D))\ F \ll_{\rm EM} G \ \&\ I \subseteq F^* \subseteq G^* \subseteq J.$$

(iii) *For $F \in \mathcal{P}_{fin}(D)$ and $I \in \mathbf{C}D$, $F \in I \Leftrightarrow F^* \ll I$.*

(iv) *For $F, G \in \mathcal{P}_{fin}(D)$, $F^* \sqsubseteq G^*$ in $\mathbf{C}D \Leftrightarrow F \sqsubseteq_{\rm EM} G$*

Here is how we extend continuous maps on D to ones on $\mathbf{C}D$.

Definition 6.1.7 If $f : D \to D$ is monotone, we extend it to a mapping

$$\bar{f} : \mathbf{C}D \to \mathbf{C}D$$

by first defining it on a basis of $\mathbf{C}D$ by $\bar{f}(F^*) = (f(F))^*$ for $F \in \mathcal{P}_{fin}(D)$, and then on all other ideals by

$$\bar{f}(I) = \bigcup_{F \in I} \bar{f}(F^*).$$

In addition, $\mathbf{C}D$ has a union operation we will need later on.

Definition 6.1.8 The function $+ : \mathbf{C}D \times \mathbf{C}D \to \mathbf{C}D$ is given by

$$I + J = \{\, H \in \mathcal{P}_{fin}(D) \,|\, \exists\, F \in I, G \in J : H \ll_{\rm EM} F \cup G \,\}.$$

Lemma 6.1.1 *Let D be a continuous dcpo. Then*

(i) *If $f : D \to D$ is monotone, then $\bar{f} : \mathbf{C}D \to \mathbf{C}D$ is Scott continuous.*

(ii) *The operation $+ : \mathbf{C}D \times \mathbf{C}D \to \mathbf{C}D$ is Scott continuous, commutative, associative, and idempotent. For $K, L \in \mathcal{P}_{com}(D)$, $K^* + L^* = (K \cup L)^*$.*

proof To see (i), note that for $F \in \mathcal{P}_{fin}(D)$, $F^* \ll I \Leftrightarrow F \in I$. Thus, the definition of $\bar{f}$ may be recast as

$$\bar{f}(I) = \bigsqcup \{\bar{f}(F^*) : F^* \ll I, F \in \mathcal{P}_{fin}(D)\}.$$

But this is the general technique by which a monotone map defined on a basis is extended to a Scott continuous map on the entire domain. (ii) is given in [7]. □

6.2 The Necessity of Lebesgue Measurements

In Chapter 5, Lebesgue measurements were introduced, and it was shown that their kernels capture the topological notion of metrizability. But where do Lebesgue measurements come from? And more importantly, what are they? In this section we answer both of these questions: They are exactly the measurements which extend to the convex powerdomain.

Definition 6.2.1 Let $\mu : D \to [0,\infty)^*$ be a monotone map on a continuous dpco. We first extend it to the abstract basis $(\mathcal{P}_{\mathit{fin}}(D), \ll_{\text{EM}})$ via

$$\mu_f : \mathcal{P}_{\mathit{fin}}(D) \to [0,\infty)^*$$

$$F \mapsto \max\{\mu x : x \in F\}$$

and then to the convex powerdomain $\mathbf{C}D$ by

$$\bar{\mu} : \mathbf{C}D \to [0,\infty)^*$$

$$I \mapsto \inf\{\mu_f(F) : F \in I\}$$

When we speak of a measurement μ extending to $\mathbf{C}D$, we mean that the mapping $\bar{\mu}$ is a measurement.

Lemma 6.2.1 *If $\mu : D \to [0,\infty)^*$ is Scott continuous on a domain, then*

(i) *The map $\bar{\mu} : \mathbf{C}D \to [0,\infty)^*$ is Scott continuous.*

(ii) *For all $F \in \mathcal{P}_{\mathit{fin}}(D)$, $\bar{\mu}(F^*) = \mu_f(F)$.*

(iii) *If $K \in \mathcal{P}_{com}(\ker\mu)$, then $\bar{\mu}(K^*) = 0$*

proof (i) If μ is monotone, then μ_f is monotone on an abstract basis. The map $\bar{\mu}$ is defined by $\bar{\mu}(I) = \bigsqcup \mu_f(I)$. Thus, it is the greatest Scott continuous map on $\mathbf{C}D$ satisfying $\bar{\mu}(F^*) \sqsubseteq \mu_f(F)$ for all $F \in \mathcal{P}_{\mathit{fin}}(D)$. This technique works for any abstract basis; the details may be found in [1].

(ii) Let $F \in \mathcal{P}_{\mathit{fin}}(D)$ and choose $x \in F$ with $\mu x = \mu_f(F)$. From (i), we have $\mu_f(F) \leq \bar{\mu}(F^*)$. Now let $n \geq 1$ be arbitrary. For each $x_i \in F$, use the continuity of μ to choose $a_i \ll x_i$ with

$$\mu x_i \leq \mu a_i < \mu x + 1/n,$$

which is possible since $\mu x_i \leq \mu x$. Then for the finite set $G_n = \{a_i : x_i \in F\}$, we see that $G_n \ll_{\text{EM}} F$, which gives

$$\bar{\mu}(F^*) \leq \mu_f(G_n) < \mu x + 1/n = \mu_f(F) + 1/n,$$

and hence $\bar{\mu}(F^*) \leq \mu_f(F)$.

(iii) Let $n \geq 1$ be fixed. For each $k \in K$, there is $a_k \ll k$ with $\mu a_k < 1/n$. Then a finite number of the a_k cover K by compactness. This yields a finite set F with $F \ll_{\text{EM}} K$ and $\mu_f(F) < 1/n$. Hence, $\bar{\mu}(K^*) < 1/n$ for each $n \geq 1$. $\Box$

Naturally, we now wonder when it is that $\bar{\mu}$ is a measurement on $\mathbf{C}D$. Before we can answer this, we need an important lemma.

Lemma 6.2.2 *Let $\mu : D \to [0, \infty)^*$ be a Lebesgue measurement on a continuous dcpo. Suppose that $F \in \mathcal{P}_{fin}(D)$ and $K \in \mathcal{P}_{com}(\ker \mu)$ with $F \ll_{\text{EM}} K$. Then there is $\lambda > 0$ such that for every $G \in \mathcal{P}_{fin}(D)$,*

$$G \ll_{\text{EM}} K \;\&\; \mu_f(G) < \lambda \Rightarrow F \ll_{\text{EM}} G$$

proof For each $x_i \in F$, choose $k_i \in K$ with $x_i \ll k_i$. Because μ is a measurement, $(\exists\, \varepsilon_i > 0)\; k_i \in \mu_{\varepsilon_i}(k_i) \subseteq \Uparrow\! x_i$. In addition, μ is a Lebesgue measurement and $K \subseteq \Uparrow\! F$ is compact, so

$$(\exists \delta > 0)(\forall k \in K)\; k \in \mu_\delta(k) \subseteq \Uparrow\! F.$$

Let $0 < \lambda < \min(\{\varepsilon_i : x_i \in F\} \cup \{\delta\})$. If $G \in \mathcal{P}_{fin}(D)$ with $G \ll_{\text{EM}} K$ and $\mu_f(G) < \lambda$, then we claim that $F \ll_{\text{EM}} G$.

To see that $F \ll_{\text{L}} G$, let $x_i \in F$. Then we know $x_i \ll k_i \in K$. Since $G \ll_{\text{EM}} K$, there is $y \in G$ with $y \ll k_i$. Because $\mu y \leq \mu_f(G) < \lambda < \varepsilon_i$, we see that $y \in \mu_{\varepsilon_i}(k_i) \subseteq \Uparrow\! x_i$, which gives $x_i \ll y \in G$.

For $F \ll_{\text{U}} G$, let $y \in G$. Since $G \ll_{\text{EM}} K$, there is $k \in K$ with $y \ll k$. Then $\mu y \leq \mu_f(G) < \lambda < \delta$, so $y \in \mu_\delta(k) \subseteq \Uparrow\! F$. Hence, there is $x \in F$ with $x \ll y$. $\Box$

Theorem 6.2.1 *For a Scott continuous $\mu : D \to [0,\infty)^*$ on a continuous dcpo D, the following are equivalent:*

(i) *The mapping μ is a Lebesgue measurement.*

(ii) *The canonical extension of μ to the convex powerdomain*

$$\bar{\mu} : \mathbf{C}D \to [0,\infty)^*$$

is a measurement.

In either case, $\ker\bar{\mu} = \{K^* : K \in \mathcal{P}_{com}(\ker\mu)\}$.

proof (i) $\Rightarrow$ (ii) Let $A \ll I$ in $\mathbf{C}D$ with $\bar{\mu}(I) = 0$. By the directedness of I and continuity of $\bar{\mu}$, there is a sequence of finite sets (F_n) such that

$$F_n \in I,\ \mu_f(F_n) < 1/n \ \&\ F_n \ll_{\text{EM}} F_{n+1},$$

for all $n \geq 1$. Set $K = \bigcap_{n\geq 1} \uparrow F_n$. This set is nonempty and compact by the Hofmann-Mislove Theorem. In addition, notice that we also have $F_n \ll_{\text{EM}} K \subseteq \ker\mu$ for all $n \geq 1$.

First we prove that $I \subseteq K^*$. Let $F \in I$ be arbitrary. Using the directedness of I, choose $M_1 \in I$ with $F, F_1 \ll_{\text{EM}} M_1$, and given any M_n, choose $M_{n+1} \in I$ with $M_n, F_{n+1} \ll_{\text{EM}} M_{n+1}$. Let $M = \bigcap \uparrow M_n$ and notice again that $M \in \mathcal{P}_{com}(\ker\mu)$. Because $F \ll_{\text{EM}} M$, $F_n \ll_{\text{EM}} M$ for all $n \geq 1$, and $\mu_f(F_n) \to 0$, Lemma 6.2.2 implies that $F \ll_{\text{EM}} F_i$, for all i sufficiently large. But we also know that $F_i \ll_{\text{EM}} K$, so transitivity of $\ll_{\text{EM}}$ gives $F \in K^*$. Hence $I \subseteq K^*$.

Finally, $\bar{\mu}$ is a measurement. Since $A \ll I$, there are $F, G \in \mathcal{P}_{fin}(D)$ with $F \ll_{\text{EM}} G$ and $A \subseteq F^* \subseteq G^* \subseteq I$, by Proposition 6.1.2. Because $F \in G^* \subseteq K^*$, we have $F \ll_{\text{EM}} K$. Using Lemma 6.2.2, choose a $\lambda > 0$ with respect to $F \ll_{\text{EM}} K$. We will prove that

$$I \in \bar{\mu}_\lambda(I) \subseteq \Uparrow A.$$

If $J \subseteq I$ and $\bar{\mu}(J) < \lambda$, there is an $H \in J$ with $\mu_f(H) < \lambda$. But then we see $H \in I \subseteq K^*$ and $\mu_f(H) < \lambda$, so by the choice of λ, $F \ll_{\text{EM}} H$. Hence,

$$A \subseteq F^* \subseteq H^* \subseteq J \ \&\ F \ll_{\text{EM}} H,$$

which gives $J \in \Uparrow A$. Thus, $\bar{\mu}$ is a measurement.

(ii) $\Rightarrow$ (i) Let $K \subseteq \ker\mu$ be Scott compact and $U \subseteq D$ be Scott open with $K \subseteq U$. By the compactness of K, there is a finite set $F \subseteq U$ with $K \subseteq \Uparrow F$ and $F \ll_{\text{EM}} K$. Thus, $F^* \ll K^*$, using Proposition 6.1.2(iii). By Lemma 6.2.1(iii), $K^* \in \ker\bar{\mu}$, and since $\bar{\mu}$ is a measurement, there is a $\lambda > 0$ with

$$K^* \in \bar{\mu}_\lambda(K^*) \subseteq \Uparrow F^*.$$

We claim that $k \in \mu_\lambda(k) \subseteq U$, for all $k \in K$.

First suppose that $k \in K$ and $x \ll k$ with $\mu x < \lambda$. By compactness of K and continuity of μ, there is a finite set G with $x \in G$, $\mu_f(G) < \lambda$ and $G \ll_{\text{EM}} K$. But then

$$G^* \ll K^* \ \& \ \bar{\mu}(G^*) = \mu_f(G) < \lambda,$$

which means $F^* \ll G^*$. Hence, $F \sqsubseteq_{\text{EM}} G$, by Proposition 6.1.2(iv). Thus, there is a $y \in F$ with $y \sqsubseteq x$ since $x \in G$. Then $x \in \uparrow F \subseteq U$

In general, if $x \in \mu_\lambda(k)$, use the directedness of $\Downarrow x$ and continuity of μ to choose $a \ll x$ with $\mu a < \lambda$. By the previous argument, $a \in U$, and since U is an upper set, $x \in U$. Hence, $k \in \mu_\lambda(k) \subseteq U$, for all $k \in K$, which means μ is a Lebesgue measurement.

Now we calculate $\ker\bar{\mu}$. The inclusion $\{K^* : K \in \mathcal{P}_{com}(\ker\mu)\} \subseteq \ker\bar{\mu}$ is clear by Lemma 6.2.1(iii). For the other, suppose that $\bar{\mu}(I) = 0$. Then as in the proof of (i) $\Rightarrow$ (ii), there is a compact $K \subseteq \ker\mu$ with $I \subseteq K^*$. But $\bar{\mu}$ is a measurement, so $I \in \max \mathbf{C}D$. Hence $I = K^*$. This finishes the proof. □

In fact, the relationship between the kernel of $\bar{\mu}$ and the compact subsets of $\ker\mu$ is much stronger than the last theorem shows.

Proposition 6.2.1 *If $\mu : D \to [0,\infty)^*$ is a Lebesgue measurement on a continuous dcpo, then the correspondence*

$$\mathcal{P}_{com}(\ker\mu) \to \ker\bar{\mu}$$

$$K \mapsto K^*$$

is a bijection.

proof The surjectivity is clear. Suppose $K^* = L^*$ for $L, K \in \mathcal{P}_{com}(\ker\mu)$. Let $x \in K$ and use μ to choose an increasing sequence (x_n) with $x_n \ll x$ and $\mu x_n < 1/n$. By the compactness of K, for each $n \geq 1$, there is a finite set F_n with $x_n \in F_n$ and $F_n \ll_{\text{EM}} K$. Then $F_n \in L^*$ so

$$(\forall n \geq 1)(\exists\, a_n \in L)\, x_n \ll a_n.$$

As the sequence (x_n) is increasing, it is clear that $a_n \to x$ in $\ker \mu$. But $\ker \mu$ is metrizable by Theorem 5.4.3, so $L \subseteq \ker \mu$ being compact must also be closed, which puts $\lim a_n = x \in L$. Thus, $K \subseteq L$. The same argument proves $L \subseteq K$. $\Box$

Thus, if a domain D models a space X, then the convex powerdomain $\mathbf{C}D$ models the *compact subsets* of X.

6.3 Contractions on the Convex Powerdomain

We apply the theory on the convex powerdomain to prove that contractions on domains extend to contractions on the convex powerdomain.

Definition 6.3.1 Let D be a continuous dcpo with a measurement μ. A monotone map $f : D \to D$ is a *contraction* if there is a constant $c < 1$ with

$$\mu f(x) \leq c \cdot \mu x$$

for all $x \in D$. The constant c is called a Lipschitz constant.

In this section we assume that D is a continuous dcpo with a Lebesgue measurement μ. Its convex powerdomain $\mathbf{C}D$ then carries the measurement $\bar{\mu}$ studied in the last section.

Proposition 6.3.1 *If $f, g : D \to D$ are contractions with respect to μ, then*

$$h : \mathbf{C}D \to \mathbf{C}D$$

$$hx = \bar{f}x + \bar{g}x$$

is a contraction with respect to $\bar{\mu}$.

proof Let f and g have Lipschitz constants c_f and c_g, respectively. We will show that h has Lipschitz constant $\max\{c_f, c_g\}$. First suppose $F \in \mathcal{P}_{fin}(D)$. By Lemma 6.1.1,

$$h(F^*) = \bar{f}(F^*) + \bar{g}(F^*) = (f(F))^* + (g(F))^* = (f(F) \cup g(F))^*,$$

which enables the estimate

$$\begin{aligned}
\bar{\mu}(h(F^*)) &= \bar{\mu}(f(F) \cup g(F))^* \\
&= \max(\mu f(F) \cup \mu g(F)) \\
&\leq \max\{c_f, c_g\} \cdot \max \mu(F) \\
&= \max\{c_f, c_g\} \cdot \bar{\mu}(F^*)
\end{aligned}$$

where the second and third equalities follow from Lemma 6.2.1(ii). Now let $I \in \mathbf{C}D$ be arbitrary. If $F \in I$, then $F^* \sqsubseteq I$, which gives

$$\bar{\mu}(h(I)) \leq \bar{\mu}(h(F^*)) \leq \max\{c_f, c_g\} \cdot \bar{\mu}(F^*) = \max\{c_f, c_g\} \cdot \mu_f(F),$$

and so by the definition of $\bar{\mu}$, $\bar{\mu}(h(I)) \leq \max\{c_f, c_g\} \cdot \bar{\mu}(I)$. $\square$

The contraction theorem (Theorem 5.1.1) may now be applied to $\bar{f}+\bar{g}$. This makes proving fixed point theorems for $\mathbf{C}D$ very simple. As an example, we consider the case which arises most often: A domain D with $\downarrow x \cap \downarrow y \neq \emptyset$ for all $x, y \in D$.

Proposition 6.3.2 *Let D be a continuous dcpo such that*

$$(\forall\, x, y \in D\,)(\,\exists\, z \in D\,)\; z \sqsubseteq x, y.$$

If $f : D \to D$ and $g : D \to D$ are contractions, and there is a point $x \in D$ with $x \sqsubseteq f(x)$ and $x \sqsubseteq g(x)$, then

$$h : \mathbf{C}D \to \mathbf{C}D$$

$$hx = \bar{f}x + \bar{g}x$$

has a unique fixed point $\mathrm{fix}(h)$ *given by*

$$\mathrm{fix}(h) = \bigsqcup_{n\geq 0} h^n\{x\}^* \in \ker \bar{\mu}.$$

Thus, there is a unique nonempty compact set $K \subseteq \ker \mu$ with $\mathrm{fix}(h) = K^*$. *In addition, for every nonempty compact $C \subseteq \ker \mu$,*

$$h^n(C^*) \to K^*$$

in the relative Scott topology on $\ker \bar{\mu}$.

proof First we prove that $\mathbf{C}D$ has the same property assumed of D. Let $I, J \in \mathbf{C}D$ and $F \in I, G \in J$. The set $F \cup G$ is finite, and so by induction, there is $z \in D$ with $\{z\} \sqsubseteq_{\mathrm{EM}} F$ and $\{z\} \sqsubseteq_{\mathrm{EM}} G$. Thus,

$$\{z\}^* \sqsubseteq F^* \sqsubseteq I \text{ and } \{z\}^* \sqsubseteq G^* \sqsubseteq J.$$

Second, we see that $\{x\}^* \sqsubseteq h\{x\}^*$, by first noting $h\{x\}^* = \{f(x), g(x)\}^*$, and then $\{x\} \sqsubseteq_{\mathrm{EM}} \{f(x), g(x)\}$ hence $\{x\}^* \sqsubseteq \{f(x), g(x)\}^* = h\{x\}^*$.

Finally, because h is contracting with respect to $\bar{\mu}$, everything follows from Theorem 5.1.1. It is worth pointing out that we have made implicit use of the bijection between $\ker \bar{\mu}$ and $\mathcal{P}_{com}(\ker \mu)$ (Proposition 6.2.1). $\Box$

Notice that $\mathbf{B}X$ and $\mathbf{U}X$ are domains with the property above that in general do not have $\perp$. That is, the condition is quite general. Of course, if a domain is pointed, we have the following.

Corollary 6.3.1 *Let D be a continuous dcpo with least element $\perp$. If f and g are contractions on D, then $\bar{f} + \bar{g} : \mathbf{C}D \to \mathbf{C}D$ has a unique fixed point given by*

$$\bigsqcup_{n \geq 0} (\bar{f} + \bar{g})^n \{\perp\}^*.$$

In addition, the conclusions of Proposition 6.3.2 also hold.

We now turn to *one possible* interpretation of Proposition 6.3.2.

6.4 An Application to Fractals

Though contractions are only monotone, they are still the extensions of continuous mappings on $\ker \mu$.

Lemma 6.4.1 *If D is a continuous dcpo with a measurement μ, then the restriction of a contraction $f : D \to D$ to $\ker \mu$ is a continuous selfmap with respect to the relative Scott topology.*

proof Let $c < 1$ be a Lipschitz constant for f. Suppose $y \ll f(x)$ with $x \in \ker \mu$. Then because f is a contraction, $f(x) \in \ker \mu$, and since μ is a measurement,

$$(\exists\, \varepsilon > 0\,)\; f(x) \in \mu_\varepsilon(f(x)) \subseteq \mathord{\uparrow\!\!\uparrow} y.$$

By the continuity of μ, choose $a \ll x$ with $\mu a < \varepsilon$. We claim that

$$x \in \mathord{\uparrow\!\!\uparrow} a \cap \ker \mu \subseteq f^{-1}(\mathord{\uparrow\!\!\uparrow} y \cap \ker \mu).$$

By the monotonicity of f, we have $f(a) \sqsubseteq f(x)$, while the contractivity of f gives $\mu f(a) \leq c \cdot \mu a < \mu a < \varepsilon$. Hence, $f(a) \in \mu_\varepsilon(f(x)) \subseteq \mathord{\uparrow\!\!\uparrow} y$. Appealing to the monotonicity of f now finishes the proof. $\Box$

We assume for the remainder of this section that D is a continuous dcpo with a Lebesgue measurement μ.

Proposition 6.4.1 *If $f : D \to D$ is a contraction with respect to μ, then*

$$\bar{f}(K^*) = (f(K))^*,$$

for all nonempty compact subsets $K \subseteq \ker \mu$.

proof First, if $K \in \mathcal{P}_{com}(\ker \mu)$, then $f(K) \in \mathcal{P}_{com}(\ker \mu)$, by the continuity of $f|_{\ker \mu}$ given in Lemma 6.4.1. Thus, $(f(K))^*$ is an element of $\mathbf{C}D$.

Now we show $\bar{f}(K^*) \sqsubseteq (f(K))^*$. If $G \in \bar{f}(K^*)$, then

$$G \in \bar{f}(K^*) = \bigcup_{F \in K^*} (f(F))^*,$$

and so there is $F \in \mathcal{P}_{fin}(D)$ with $F \ll_{\text{EM}} K$ and $G \ll_{\text{EM}} f(F)$. Then $G \ll_{\text{EM}} f(F) \sqsubseteq_{\text{EM}} f(K)$ which gives $G \ll_{\text{EM}} f(K)$ and hence $G \in (f(K))^*$.

Finally, $\bar{f}$ is a contraction on $\mathbf{C}D$, by Prop. 6.3.1 (applied with $f = g$), and $K^* \in \ker \bar{\mu}$, by Theorem 6.2.1, so $\bar{f}(K^*) \in \ker \bar{\mu} \subseteq \max \mathbf{C}D$. Thus, $\bar{f}(K^*) = (f(K))^*$ $\Box$

The view that contractions are extensions of continuous maps on $\ker \mu$ leads to the following.

Theorem 6.4.1 *Let D be a continuous dcpo such that*

$$(\,\forall x, y \in D\,)(\,\exists z \in D\,)\; z \sqsubseteq x, y.$$

If $f : D \to D$ and $g : D \to D$ are contractions for which

$$(\,\exists x \in D\,)\; x \sqsubseteq f(x) \;\&\; x \sqsubseteq g(x),$$

then there is a unique $K \in \mathcal{P}_{com}(\ker \mu)$ such that $f(K) \cup g(K) = K$.

proof By Proposition 6.3.2, the map $h : \mathbf{C}D \to \mathbf{C}D$ given by $hx = \bar{f}x + \bar{g}x$ has a unique fixed point $\text{fix}(h) = K^*$, where $K \in \mathcal{P}_{com}(\ker \mu)$. Now observe that for any $C \in \mathcal{P}_{com}(\ker \mu)$,

$$h(C^*) = \bar{f}(C^*) + \bar{g}(C^*) = (f(C))^* + (g(C))^* = (f(C) \cup g(C))^*,$$

where the second equality follows from Prop. 6.4.1. Then since $h(K^*) = K^*$, we have $(f(K) \cup g(K))^* = K^*$, which gives $f(K) \cup g(K) = K$, using the bijection of Prop. 6.2.1.

If $C \in \mathcal{P}_{com}(\ker \mu)$ satisfies $f(C) \cup g(C) = C$, then $h(C^*) = C^*$, which by the uniqueness of K^* gives $K^* = C^*$. But then once again Prop. 6.2.1 yields $K = C$. $\Box$

Corollary 6.4.1 *If $f : D \to D$ and $g : D \to D$ are contractions on a domain with least element $\bot$, then there is a unique $K \in \mathcal{P}_{com}(\ker \mu)$ such that $f(K) \cup g(K) = K$.*

We now consider a natural application of these results.

Definition 6.4.1 An *iterated function system* (IFS) on a space X is a nonempty finite collection of continuous selfmaps on X. We denote an IFS by $(X; f_1, \ldots, f_n)$.

Definition 6.4.2 An IFS $(X; f_1, \ldots, f_n)$ is *hyperbolic* if X is a complete metric space and f_i is a contraction for all $1 \leq i \leq n$.

Hyperbolic iterated function systems are used to model fractals: Given a fractal image, one searches for a hyperbolic IFS which models it. But what does it mean to model an *image*? The answer is given by Hutchinson's fundamental result [12].

Theorem 6.4.2 (Hutchinson) *If $(X; f_1, \ldots, f_n)$ is a hyperbolic IFS on a complete metric space X, then there is a unique nonempty compact subset $K \subseteq X$ such that*

$$K = \bigcup_{i=1}^{n} f_i(K).$$

In [6], Abbas Edalat used the upper space $\mathbf{U}X$ to give a domain theoretic proof of Theorem 6.4.2 in the special case of a compact metric space X. We can give a simpler proof by using Theorem 6.4.1.

Example 6.4.1 If we have two contractions $f, g : X \to X$ on a compact metric space X, they have extensions

$$\bar{f}, \bar{g} : \mathbf{U}X \to \mathbf{U}X$$

which are contractions on $\mathbf{U}X$ with respect to $\lambda = \text{diam}$. But λ is a Lebesgue measurement on a domain $\mathbf{U}X$ with bottom element $\bot = X$. Thus,

$$(\,\exists! K \in \mathcal{P}_{com}(\ker \lambda)\,)\ \bar{f}(K) \cup \bar{g}(K) = K,$$

by the Corollary to Theorem 6.4.1. Because $\ker \lambda \simeq X$ and the mappings $\bar{f}, \bar{g}$ extend f and g, it is clear that

$$(\,\exists! K \in \mathcal{P}_{com}(X)\,)\ f(K) \cup g(K) = K,$$

which finishes the proof.

In [7], Edalat and Heckmann used the formal ball model $\mathbf{B}X$ to give a domain theoretic proof of Theorem 6.4.2 for any complete metric space X. We can give a simpler proof by using Theorem 6.4.1.

Example 6.4.2 If we have two contractions $f, g : X \to X$ on a complete metric space X, they have extensions

$$\bar{f}, \bar{g} : \mathbf{B}X \to \mathbf{B}X$$

which are contractions on $\mathbf{B}X$ with respect to $\pi(x, r) = r$. But π is a Lebesgue measurement on a domain which has the property that for all $(x, r), (y, s) \in \mathbf{B}X$, there is an element $z = (x, r + s + d(x, y)) \in \mathbf{B}X$ with $z \sqsubseteq (x, r), (y, s)$. In addition, for any $x \in X$, choosing r so that

$$r \geq \frac{d(x, fx)}{1 - c_f} \text{ and } r \geq \frac{d(x, gx)}{1 - c_g},$$

where $c_f, c_g < 1$ are the Lipschitz constants for f and g, respectively, gives a point $(x, r) \sqsubseteq \bar{f}(x, r), \bar{g}(x, r)$. By Theorem 6.4.1,

$$(\, \exists! K \in \mathcal{P}_{com}(\ker \pi) \,)\; \bar{f}(K) \cup \bar{g}(K) = K.$$

However, because $\ker \pi \simeq X$ and the mappings $\bar{f}, \bar{g}$ extend f and g, it is clear that

$$(\, \exists! K \in \mathcal{P}_{com}(X) \,)\; f(K) \cup g(K) = K,$$

which finishes the proof.

By now the reader can surely appreciate the relevance of the work done in Chapter 5. For example, if a space may be realized as the kernel of a Lebesgue measurement on a continuous *dcpo* D, then Theorem 6.4.1 implies that Hutchinson's result holds for any finite family of contractions which extend to D. So we ask again, which spaces may be realized as the kernels of Lebesgue measurements on continuous dcpo's?

Certainly all complete metric spaces may be realized in this manner, and in the other direction, we know that any such space must be metrizable by Theorem 5.4.3. But the metrizability holds even at the level of a continuous poset. Does imposing directed completeness on a continuous poset now force the kernel of a Lebesgue measurement to be *completely metrizable,* or have we found a way of generalizing Hutchinson's theorem?

6.5 The Topological Completeness of Domains

It is obvious that domains are topologically complete. What is not obvious is how to prove this, since in their Scott topologies, they are only T_0 spaces.

Definition 6.5.1 Let (X, τ) be a space and $\tau_* = \{(U, x) : x \in U \in \tau\}$. (X, τ) is *Choquet complete* if there is a sequence $(a_n)_{n \geq 1}$ of functions

$$a_n : \tau_*^n \to \tau$$

such that

(i) For each $((U_1, x_1), \cdots, (U_n, x_n)) \in \mathrm{dom}(a_n)$,

$$x_n \in a_n((U_1, x_1), \cdots, (U_n, x_n)) \subseteq U_n,$$

and

(ii) For any sequence (V_n, x_n) in τ_* with $V_{n+1} \subseteq a_n((V_1, x_1), \cdots, (V_n, x_n))$, for all $n \geq 1$, we have

$$\bigcap_{n \geq 1} V_n \neq \emptyset.$$

Notice that the function a_n maps nonempty open sets to nonempty open sets. Choquet complete spaces possess abstract notions of the two fundamentals of computation: (i) *approximation* and (ii) *completeness.* The next result appears in [5].

Theorem 6.5.1 (Choquet) *The following both hold:*

(i) *A locally compact Hausdorff space is Choquet complete.*

(ii) *A metric space is Choquet complete iff it is completely metrizable.*

Choquet completeness is inherited by G_δ sets [14] – a property any purported notion of completeness ought to have.

Proposition 6.5.1 *A G_δ subset of a Choquet complete space is Choquet complete.*

Corollary 6.5.1 *A Čech-complete space is Choquet complete.*

proof A Čech-complete space is a G_δ subset of a compact Hausdorff space. Now apply Theorem 6.5.1(i) followed by Prop. 6.5.1. $\Box$

However, unlike Čech-completeness, which requires spaces to be Tychonoff, Choquet completeness requires no separation whatsoever. The consequences of this are fundamental.

Theorem 6.5.2 *The Scott topology on a continuous dcpo D is Choquet complete.*

proof We define the approximation scheme

$$a : \{(U, x) : x \in U \in \sigma_D\} \to \sigma_D$$

as follows: Given a Scott open set U and a point $x \in U$, there is $b \in U$ with $b \ll x$. By interpolation, there is z with $b \ll z \ll x$. Set $a(U, x) = \Uparrow z$ and note that

$$x \in a(U, x) = \Uparrow z \subseteq \uparrow b \subseteq U.$$

In this way, we have defined a so that for all $(U, x) \in \mathrm{dom}(a)$, there is a Scott compact upper set K with $x \in a(U, x) \subseteq K \subseteq U$.

Finally, given elements $(U_n, x_n) \in \mathrm{dom}(a)$ with $U_{n+1} \subseteq a(U_n, x_n) \subseteq U_n$, for all $n \geq 1$, we immediately obtain a decreasing sequence of nonempty Scott compact upper sets (K_n) with

$$\bigcap_{n\geq 1} U_n = \bigcap_{n\geq 1} K_n.$$

But this intersection is nonempty by the Hofmann-Mislove theorem.

Setting $a_n((U_1, x_1), \cdots, (U_n, x_n)) = a(U_n, x_n)$ finishes the proof. $\Box$

And so this is the sense in which domains are topologically complete: The same sense in which complete metric spaces and locally compact Hausdorff spaces are complete.

Corollary 6.5.2 *The Scott topology on a continuous dcpo is Baire.*

proof This is true of any Choquet complete space. $\Box$

Proposition 6.5.2 *If a metric space is embedded in a continuous dcpo as a G_δ subset, then it is completely metrizable.*

proof Let X be a metric space which embeds in a continuous dcpo D as a G_δ subset. By Theorem 6.5.2, the Scott topology on D is Choquet complete. Thus, X is Choquet complete by Prop. 6.5.1. By Theorem 6.5.1(ii), X is completely metrizable. □

Finally, we are able to observe the effect that directed completeness has on the kernels of Lebesgue measurements.

Theorem 6.5.3 *Let X be a topological space.*

(i) *X is metrizable iff it is the kernel of a Lebesgue measurement on a continuous poset.*

(ii) *X is completely metrizable iff it is the kernel of a Lebesgue measurement on a continuous dcpo.*

In view of Theorem 5.5.1, we see that the complete metrizability in (ii) above can be represented by a measurement.

Corollary 6.5.3 *If μ is a Lebesgue measurement on a continuous dcpo D, there is a complete Lebesgue measurement λ on D such that $\mu \leq \lambda$ and* $\ker \mu = \ker \lambda$.

Thus, if we use $\bar{\mu}$ as the preferred method for extending a measurement μ to $\mathbf{C}D$, then $\ker \mu$ must be completely metrizable, and so the answer to our question is no: We cannot extend Hutchinson's theorem using $\bar{\mu}$. It is up to the reader to decide whether or not this represents a failure of the approach, or if the seeming naturality of $\bar{\mu}$ reveals something rather deep: That complete metric spaces are *necessary* if one wants to prove Hutchinson's theorem within a computational framework.

6.6 Summary

At this point the entire chapter may seem rather silly. Why do all this work to prove something we already knew? That is *not* why we did it.

Contractions on domains are elegant methods for calculating one member of the kernel. They arise in various contexts and more often than not, they are not recognized as such by the people using them. For example, consider the number of times that the following example has been quoted in theoretical computer science:

$$\phi : [\mathbb{N} \to \mathbb{N}] \to [\mathbb{N} \to \mathbb{N}]$$

$$\phi(f)(k) = \begin{cases} 1 & \text{if } k = 0, \\ kf(k-1) & \text{if } k \geq 1 \ \& \ k-1 \in \text{dom}\, f. \end{cases}$$

The author was told once that there was nothing new which could be said about this example. Of course, there is always something new which can be said about things, this is especially the case with simple things that have been examined time and time again. And indeed, in Chapter 2, we did say something new about it: It is a contraction on a domain.

Now one might argue that measurements were not around before, so how could anyone have noticed that ϕ was a contraction? And that is a good argument. Precisely. They were not around. And so we have an example of how it is that a certain viewpoint, a certain philosophy, can have the magical effect of enabling us to see things we may not have been able to see otherwise.

And what follows from this *one* observation about ϕ? That it has a unique fixed point calculable by iterating on $\perp$. That this unique fixed point is maximal. And these are good things. But we also know that ϕ must be the extension of a continuous selfmap on $\max[\mathbb{N} \to \mathbb{N}]$, and that on this subspace, its unique fixed point is an attractor. In addition, since $\max[\mathbb{N} \to \mathbb{N}] = \ker|\text{dom}|$ is the kernel of a Lebesgue measurement on a countably based domain, such a space is Polish. Then, *should we choose to,* we may say that ϕ is nothing more than the extension to a domain of a contraction on the Polish space $\max[\mathbb{N} \to \mathbb{N}] \simeq \mathbb{R} \setminus \mathbb{Q}$. This is an example of the kind of viewpoint which *blurs* the vision.

So while we like knowing that ϕ is a contraction on a domain, we don't care at all that it is the extension of a contraction on a Polish space. In a similar way, we may only care that contractions on domains extend to contractions on $\mathbf{C}D$, and may not at all be interested in the fact that these contractions correspond to hyperbolic iterated function systems – but the theory is there should the need arise.

Then this chapter says the following: Anytime contractions on a domain are used to model something, we automatically know how they behave on the convex powerdomain. In addition, we may view them as hyperbolic IFS's *if we want to.*

6.7 Questions

We have left a number of important questions for future work.

(i) Characterize $\max \mathbf{C}D$. Of interest is the case when $\ker \mu = \max D$.

(ii) Prove that the relative Scott topology on $\ker \bar{\mu}$ is homeomorphic to $\mathcal{P}_{com}(\ker \mu)$ with the topology induced by the Hausdorff metric (derived from a metric on $\ker \mu$).

(iii) Prove that $\bar{\mu}$ is a Lebesgue measurement iff it is a measurement.

(iv) How do we extend splittings $s : D \to D$ to splittings $\bar{s}$ on $\mathbf{C}D$. In particular, we want such extensions to satisfy $\bar{s}(F^*) = (s(F))^*$.

(v) What is a measurement on an abstract basis?

Question (ii) is of particular importance: Unless it is answered, our analysis of hyperbolic IFS's, as well as Edalat's, will remain incomplete.

Chapter 7

The Informatic Derivative

An algorithm is an example of a function which constructively manipulates information. As we saw with the renee equation, we may think of it as an iterative process. At each step of this iteration, we can measure the amount of information it intends to manipulate next. In this way, we may be able to predict whether or not it is heading toward a solution. At the same time, because we can keep track of how information content fluctuates during this process, it is reasonable to expect that we can determine the *rate* at which it does so.

7.1 Sets and Functions

In this chapter we shall be concerned with *partial functions* $f : D \to \mathbb{R}$ defined on a subset $\mathrm{dom}(f)$ of a continuous dcpo D. The set $\mathrm{dom}(f)$ will be regarded a space in its relative μ topology, while the real line will carry its usual topology.

7.2 Limits and Continuity

Definition 7.2.1 Let X and Y be topological spaces and $A \subseteq X$. For a function $f : A \to Y$ and a point $p \in X$, we write

$$\lim_{x \to p} f(x) = y$$

if for all open sets $V \subseteq Y$ around y, there is an open set $U \subseteq X$ around p, such that $f(U \cap A) \subseteq V$

If Y is a Hausdorff space and $p \in \bar{A}$, then the value y is *unique* and is called *the* limit of f as x tends to p.

Definition 7.2.2 A point p in a space X is *isolated* if $\{p\}$ is an open set.

Definition 7.2.3 Let X be a space with a subset $A \subseteq X$. A point $p \in X$ is an *accumulation point* of A if for every open set $U \subseteq X$ with $p \in U$, we have $U \cap (A \setminus \{p\}) \neq \emptyset$.

To take limits of functions at points *not* in their domains, we must work with accumulation points. Notice that accumulation points are *never* isolated.

Lemma 7.2.1 *Let $f : D \to \mathbb{R}$ be a partial map on a domain and μ be a measurement on D with $\mu \to \sigma_X$. If $p \in X$ and $p \in \mathrm{Cl}_\mu(\mathrm{dom}(f)) \setminus \mathrm{dom}(f)$, then*

$$\lim_{x \to p} f(x) = L$$

iff for all $\varepsilon > 0$, there is a $\delta > 0$, such that

$$(\, x \in \mathrm{dom}(f) \text{ and } x \sqsubseteq p \text{ and } 0 < |\mu x - \mu p| < \delta \,) \Rightarrow |f(x) - L| < \varepsilon.$$

In either case, the real number L is unique.

proof ($\Rightarrow$) Given $\varepsilon > 0$, there is a μ open set $U = \mathord{\uparrow\!\!\!\uparrow} a \cap \mathord{\downarrow} p$ such that

$$p \in U \text{ and } f(U \cap \mathrm{dom}(f)) \subseteq (L - \varepsilon, L + \varepsilon).$$

Because $\mu \to \sigma_X$ and $p \in X$,

$$(\exists \delta > 0)\, x \sqsubseteq p \text{ and } |\mu x - \mu p| < \delta \Rightarrow x \in U.$$

This value of $\delta > 0$ works. For ($\Leftarrow$), any open subset of the real line around L can be replaced with one of the form $(L - \varepsilon, L + \varepsilon)$, for $\varepsilon > 0$. Let $\delta > 0$ be the value in the statement of the lemma and set $\lambda = \delta + \mu p > 0$. Then

$$f(\mathrm{dom}(f) \cap \mu_\lambda(p)) = f(\mathrm{dom}(f) \cap (\mu_\lambda(p) \setminus \{p\})) \subseteq (L - \varepsilon, L + \varepsilon),$$

where the equality follows from $p \notin \mathrm{dom}(f)$, while the inclusion uses the strict monotonicity of μ in Lemma 2.2.5 and the definition of δ.

For the uniqueness of L, we use the fact that the topology on the real line is Hausdorff, and that $p \in \mathrm{Cl}_\mu(\mathrm{dom}(f))$. $\square$

The intuitions from analysis are clear: The domain D is to be identified with the real line, the measurement μ is identified with a real variable, and a set $\mu_\varepsilon(x)$ is like an interval $(x - \varepsilon, x + \varepsilon)$.

Definition 7.2.4 A partial mapping $f : D \to E$ defined on a subset of a continuous dcpo D is μ *continuous* if it is topologically continuous as a map from $\mathrm{dom}(f)$ with its relative μ topology to E with its μ topology.

If $\mathrm{dom}(f)$ is a *subdomain,* we know its relative μ topology is the same as its intrinsic μ topology by the work of Chapter 3. In this sense, the definition above extends the definition of μ continuity between domains.

7.3 Derivatives of Real Valued Maps on Domains

Definition 7.3.1 Let D be a continuous dcpo with a measurement $\mu \to \sigma_X$. If $f : D \to \mathbb{R}$ is a partial map and $p \in \mathrm{dom}(f) \cap X$ is not isolated in $\mathrm{dom}(f)$, then

$$\frac{df}{d\mu}(p) := \lim_{x \to p} \frac{f(x) - f(p)}{\mu x - \mu p}$$

is called a *derivative* of f at p with respect to μ, whenever this limit exists in the relative μ topology on $\mathrm{dom}(f)$ and the usual topology on $\mathbb{R}$.

There are few points worth stressing about this definition. First, p is not an isolated point of $\mathrm{dom}(f)$, so in particular it can never be a compact element in D. Second, the mapping μ must induce the Scott topology on a set X containing p. Finally, when a mapping has a derivative, it is unique.

Lemma 7.3.1 *If a partial map $f : D \to \mathbb{R}$ has a derivative at a point $p \in \mathrm{dom}(f) \cap X$ with respect to $\mu \to \sigma_X$, then it is unique.*

proof Let $A = \{x \in \mathrm{dom}(f) : \mu x \neq \mu p\}$ and define $g : A \to \mathbb{R}$ by the quotient

$$g(x) = \frac{f(x) - f(p)}{\mu x - \mu p}.$$

Because p is not isolated in $\mathrm{dom}(f)$, for each $a \ll p$, there is an element $x_a \in D$ such that

$$x_a \in \mathrm{dom}(f) \cap (\twoheaduparrow a \cap \downarrow p) \text{ and } x_a \neq p.$$

By the strict monotonicity of μ in Lemma 2.2.5, $\mu x_a > \mu p$. Hence $x_a \in A$. Then we have shown that each basic μ open set around p hits A. Thus, $p \in \bar{A}$. By Lemma 7.2.1, $\lim_{x \to p} g(x)$ is unique. $\Box$

In each of the following examples, μ is the length measurement on $\mathbf{I}\mathbb{R}$.

Proposition 7.3.1 *If $f : \mathbb{R} \to \mathbb{R}$ is a function on the real line and we define*

$$\Delta f : \mathbf{I}\mathbb{R} \to \mathbb{R}$$

$$\Delta f[a, b] = f(b) - f(a),$$

then for all $p \in \mathbb{R}$,

$$\frac{d\Delta f}{d\mu}[p] \text{ exists } \Leftrightarrow \; f'(p) \text{ exists},$$

and in either case these quantities coincide.

proof Consider the case $f'(p) > 0$. For all $0 < \varepsilon < f'(p)$, there is $\delta > 0$ with

$$0 < |x - p| < \delta \Rightarrow 0 < f'(p) - \varepsilon < \frac{f(x) - f(p)}{x - p} < \varepsilon + f'(p)$$

by the definition of the classical derivative.

Let $\bar{x} = [a, b] \in \mathbf{I}\mathbb{R}$ with $\bar{x} \sqsubseteq [p]$ and $0 < \mu\bar{x} < \delta$. If $b \neq p$, then

$$(f'(p) - \varepsilon)(b - p) < f(b) - f(p) < (\varepsilon + f'(p))(b - p),$$

while for $a \neq p$ we have

$$(f'(p) - \varepsilon)(p - a) < f(p) - f(a) < (\varepsilon + f'(p))(p - a).$$

If $a \neq p$ and $b \neq p$, then adding the inequalities above gives

$$\left| \frac{f(b) - f(a)}{b - a} - f'(p) \right| < \varepsilon$$

Otherwise either $a = p$ or $b = p$, in which case the inequality above is trivial.

Then we have shown that when $f'(p) > 0$ exists, so does $\frac{d\Delta f}{d\mu}[p]$, and that these values are equal. The case $f' < 0$ follows by replacing f with $-f$, and the case $f'(p) = 0$ is trivial.

The other direction, that the existence of $\frac{d\Delta f}{d\mu}[p]$ implies that of $f'(p)$, follows by considering nontrivial intervals of the form $[x, p]$ and $[p, x]$. $\square$

Notice that Δf has the interesting property that when restricted to the real line $\mathbb{R} = \max \mathbf{I}\mathbb{R}$ it is the identically zero function. At the same time, much of the important information about the function f can be extracted from Δf domain theoretically.

Example 7.3.1 For a continuous function $f : \mathbb{R} \to \mathbb{R}$ on the real line, define

$$\mathrm{I}f : \mathbf{I}\mathbb{R} \to \mathbb{R}$$

$$\mathrm{I}f[a,b] = \int_a^b f(x)\, dx$$

Applying the mean value theorem for integrals yields

$$\frac{d(\mathrm{I}f)}{d\mu}[p] = f(p)$$

for all $p \in \mathbb{R}$.

We have seen that algorithms may be naturally viewed as maps on domains. In fact, for some algorithms, we *must* use domains if we'd like to view them as functions (bisection). The previous examples show that mappings on domains provide natural descriptions of key ideas in analysis as well.

7.4 Derivatives of Mappings Between Domains

First we consider derivatives of selfmaps.

Definition 7.4.1 Let (D, μ) be a domain with a measurement $\mu \to \sigma_X$. If $f : D \to D$ is a partial map and $p \in \mathrm{dom}(f) \cap X$ is not isolated in $\mathrm{dom}(f)$, then

$$\frac{df}{d\mu}(p) := \frac{d(\mu f)}{d\mu}(p),$$

is called the derivative of f at p with respect to μ if it exists.

We also write the derivative as $f_\mu(p)$, $df_\mu(p)$, and sometimes even as $df(p)$.

Lemma 7.4.1 *If a partial map $f : D \to D$ has a derivative at p with respect to $\mu \to \sigma_X$ and $f(p) \in X$, then the following are equivalent:*

(i) *The partial map f is μ continuous at p.*

(ii) *There is an $a \ll p$ such that*

$$(t \in \mathrm{dom}(f) \text{ and } a \ll t \sqsubseteq p) \Rightarrow f(t) \sqsubseteq f(p).$$

In particular, (ii) *is satisfied by monotone mappings.*

proof (ii) $\Rightarrow$ (i) Given $y \ll f(p)$, we need to find $a \ll p$ such that

$$(\forall t \in \mathrm{dom}(f))\, a \ll t \sqsubseteq p \Rightarrow y \ll f(t),$$

which also satisfies the property given in (ii) above. If there is no such a, we can find elements $x_n \in \mathrm{dom}(f)$ with $x_n \sqsubseteq p$, $\mu x_n \to \mu p$, $f(x_n) \sqsubseteq f(p)$, and $f(x_n) \notin \Uparrow y$. Because f has a derivative at p,

$$\lim_{n\to\infty} (\mu f(x_n) - \mu f(p)) = df_\mu(p) \cdot \lim_{n\to\infty} (\mu x_n - \mu p) = 0,$$

which means $\mu f(x_n) \to \mu f(p)$. By Proposition 3.1.6, $f(x_n) \to f(p)$ in the μ topology on D. But $\Uparrow y$ is a μ open set containing $f(p)$, so $f(x_n) \in \Uparrow y$, for most n. This is a contradiction. $\square$

The last lemma is important to keep in mind: The μ continuity of a partial map is easy to check at a point where it has a derivative.

Lemma 7.4.2 *If a partial map $f : (D, \mu) \to (D, \mu)$ has a derivative at a point p where it is μ continuous, then $df_\mu(p) \geq 0$.*

proof The derivative is defined as

$$df_\mu(p) := \lim_{x\to p} \frac{\mu f(x) - \mu f(p)}{\mu x - \mu p}.$$

By the μ continuity of f at p, there is an $a \ll p$ such that $f(x) \sqsubseteq f(p)$ for all $x \in \mathrm{dom}(f)$ with $a \ll x \sqsubseteq p$. Then by the monotonicity of μ,

$$\frac{\mu f(x) - \mu f(p)}{\mu x - \mu p} \geq 0,$$

for all $x \in \mathrm{dom}(f) \cap \Uparrow a \cap \downarrow p$ with $x \neq p$. Hence, $df_\mu(p) \geq 0$. $\square$

The derivative of the identity is 1 and the derivative of a constant map is 0.

Lemma 7.4.3 *Let D be a continuous dcpo with a measurement $\mu \to \sigma_X$. If $p \in X$ is not a compact element of D, then*

(i) *If $1 : D \to D$ is the identity map, $1_\mu(p) = 1$.*

(ii) *If $f : D \to D$ is a constant map, $df_\mu(p) = 0$.*

Example 7.4.1 Informatic Linearity.

(i) Let $\mu = |\text{dom}|$ be the natural measurement on $[\mathbb{N} \to \mathbb{N}]$. Consider the operator

$$\phi : [\mathbb{N} \to \mathbb{N}] \to [\mathbb{N} \to \mathbb{N}]$$

$$\phi(f)(k) = \begin{cases} 1 & \text{if } k = 0, \\ kf(k-1) & \text{if } k \geq 1 \ \& \ k-1 \in \text{dom}\, f. \end{cases}$$

At a total function $f \in \ker \mu = \max[\mathbb{N} \to \mathbb{N}]$, we have

$$\frac{d\phi}{d\mu}(f) = \lim_{x \to f} \frac{\mu\phi(x) - \mu\phi(f)}{\mu x - \mu f} = \lim_{x \to f} \frac{\mu\phi(x)}{\mu x} = \lim_{x \to f} \frac{\mu x/2}{\mu x} = \frac{1}{2},$$

where we have used the equality $\mu\phi = \mu/2$ obtained in Example 2.6.3.

(ii) Let $\mu = 1/2^{|\cdot|}$ be the natural measurement on Σ^∞. The shift map

$$s : \Sigma^\infty \to \Sigma^\infty$$

$$s(x) = \text{rest}(x)$$

removes the first bit from a nonempty string x and leaves ε fixed. Its derivative at an infinite string $p \in \ker \mu = \max \Sigma^\infty$ is

$$\frac{ds}{d\mu}(p) = \lim_{x \to p} \frac{\mu s(x)}{\mu x} = \lim_{x \to p} \frac{1/2^{|s(x)|}}{1/2^{|x|}} = \lim_{x \to p} \frac{1/2^{|x|-1}}{1/2^{|x|}} = 2.$$

The fact that $ds_\mu(p) > 1$ is interesting in view of the fact that the restriction of s to $\max \Sigma^\infty$ is known to be chaotic.

Our intuitive sense that the mappings of the last example are "simple" is made formally precise by showing that the rate at which they manipulate (amounts of) information is constant. Now for a more interesting example.

Definition 7.4.2 If $f : \mathbb{R} \to \mathbb{R}$ is a continuous map on the real line, it has a canonical extension to the interval domain

$$\bar{f} : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$$

$$[a, b] \mapsto f[a, b]$$

What happens if we try to differentiate $\bar{f}$ with respect to μ?

Lemma 7.4.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map which is differentiable at $p \in \mathbb{R}$ with $f'(p) > 0$. Then there is $\delta > 0$ such that for all $\bar{x} \in \mathbf{I}\mathbb{R}$ with $0 < \mu\bar{x} < \delta$ and $\bar{x} \sqsubseteq [p]$, there are numbers $m_1, m_2 \in \bar{x}$ satisfying*

$$m_1 \neq m_2,\ m_1 \leq p \leq m_2,\ f(m_1) = \min_{t \in \bar{x}} f(t),\ \textit{and}\ f(m_2) = \max_{t \in \bar{x}} f(t).$$

proof Because $f'(p) > 0$, there is $\delta > 0$ such that

$$0 < |x - p| < \delta \Rightarrow \frac{f(x) - f(p)}{x - p} > 0.$$

We claim that this value of δ works. Let $0 < \mu\bar{x} < \delta$ and $\bar{x} \sqsubseteq [p]$.

The function f is continuous on the compact set $\bar{x}$, so it has a minimum at $m_1 \in \bar{x}$ and a maximum at $m_2 \in \bar{x}$. If $m_1 = m_2$, then f is constant on $\bar{x}$. But since $\mu\bar{x} > 0$ and $p \in \bar{x}$, we must have $f'(p) = 0$. Hence, $m_1 \neq m_2$.

Finally, we show that $m_1 \leq p \leq m_2$. If $m_1 > p$, then

$$0 < |m_1 - p| \leq \mu\bar{x} < \delta \Rightarrow f(m_1) > f(p),$$

which contradicts the fact that $f(m_1)$ is the minimum value of f on $\bar{x}$. Thus, $m_1 \leq p$. Similarly, $p \leq m_2$. $\square$

As it turns out, $\bar{f}$ has a derivative whenever f does.

Theorem 7.4.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map on the real line. If f is differentiable at p, then*

$$\frac{d\bar{f}}{d\mu}[p] = |f'(p)|$$

for all $p \in \mathbb{R}$.

proof First consider the case that $f'(p) > 0$. Let $\varepsilon > 0$. Using Prop. 7.3.1 and Lemma 7.4.4, there is a $\delta > 0$ such that for all $\bar{x} \in \mathbf{I}\mathbb{R}$ with $\bar{x} \sqsubseteq [p]$ and $0 < \mu\bar{x} < \delta$, we have

$$(*) \quad \left| \frac{\Delta f(\bar{x})}{\mu\bar{x}} - f'(p) \right| < \varepsilon,$$

and that the condition of Lemma 7.4.4 is satisfied for $\bar{x}$.

By Lemma 7.4.4, there are $m_1, m_2 \in \bar{x}$ with $m_1 \neq m_2$, and f assumes its max and min values on $\bar{x}$ at these points. Thus,

$$(1)\ \frac{\mu\bar{f}(\bar{x})}{\mu\bar{x}} = \frac{f(m_2) - f(m_1)}{\mu\bar{x}} \leq \frac{f(m_2) - f(m_1)}{m_2 - m_1}.$$

On the other hand, the inequality

$$(2)\ \frac{\Delta f(\bar{x})}{\mu\bar{x}} \leq \frac{\mu\bar{f}(\bar{x})}{\mu\bar{x}}$$

is trivial.

Because $\bar{y} := [m_1, m_2]$ is an interval which satisifies $0 < \mu\bar{y} \leq \mu\bar{x} < \delta$ *and* $\bar{y} \sqsubseteq [p]$, we combine (*) and (1) to obtain

$$\frac{\mu\bar{f}(\bar{x})}{\mu\bar{x}} - f'(p) \leq \frac{\Delta f(\bar{y})}{\mu\bar{y}} - f'(p) < \varepsilon.$$

By applying (*) and (2) we get

$$-\varepsilon < \frac{\Delta f(\bar{x})}{\mu\bar{x}} - f'(p) \leq \frac{\mu\bar{f}(\bar{x})}{\mu\bar{x}} - f'(p).$$

Then we have shown

$$\left|\frac{\mu\bar{f}(\bar{x})}{\mu\bar{x}} - f'(p)\right| < \varepsilon$$

which finishes the proof in this case.

For $f'(p) < 0$, let $g = -f$. Since $g'(p) > 0$, the work just done gives

$$\frac{d\bar{g}}{d\mu}[p] = g'(p) = -f'(p) = |f'(p)|.$$

But $\mu\bar{f} = \mu\bar{g}$, so the very same thing is true of $\bar{f}$.

Finally, the case $f'(p) = 0$. Let $\varepsilon > 0$. Using the definition of the classical derivative, there is a $\delta > 0$ such that for all x with $0 < |x - p| < \delta$,

$$\left|\frac{f(x) - f(p)}{x - p}\right| < \frac{\varepsilon}{2}.$$

Let $\bar{x} \sqsubseteq [p]$ and $0 < \mu\bar{x} < \delta$. By the continuity of f, choose points $m_1, m_2 \in \bar{x}$ where f takes its max and min values on $\bar{x}$. If $m_1 \neq p$ and $m_2 \neq p$, then

$$\begin{aligned} \frac{\mu\bar{f}(\bar{x})}{\mu\bar{x}} &\leq \frac{|f(m_1) - f(p)|}{\mu\bar{x}} + \frac{|f(m_2) - f(p)|}{\mu\bar{x}} \\ &\leq \frac{|f(m_1) - f(p)|}{|m_1 - p|} + \frac{|f(m_2) - f(p)|}{|m_2 - p|} \\ &< \varepsilon. \end{aligned}$$

If we have either $m_1 = p$ or $m_2 = p$, but not both, then one of the terms in the first inequality disappears. Finally, if $m_1 = p = m_2$, then $\mu\bar{f}(\bar{x}) = 0$. Thus, $\mu\bar{f}(\bar{x})/\mu\bar{x} < \varepsilon$ always holds, which proves $\bar{f}_\mu[p] = 0$, finishing the proof. $\square$

There is a converse for the case $\bar{f}_\mu[p] = 0$.

Corollary 7.4.1 *For a continuous map $f : \mathbb{R} \to \mathbb{R}$ on the real line,*

$$\frac{d\bar{f}}{d\mu}[p] = 0 \Leftrightarrow f'(p) = 0,$$

for all $p \in \mathbb{R}$.

proof Suppose $\bar{f}_\mu[p] = 0$ and let $\varepsilon > 0$ be given. Let $\delta > 0$ be the number given in the definition of $f_\mu[p]$. Let $0 < |x - p| < \delta$. Then $x \neq p$ so these points determine an interval $\bar{x}$ with $0 < \mu\bar{x} = |x - p| < \delta$ and $\bar{x} \sqsubseteq [p]$. Then

$$\frac{|f(x) - f(p)|}{|x - p|} \leq \frac{\mu\bar{f}(\bar{x})}{\mu\bar{x}} < \varepsilon,$$

which means $f'(p) = 0$. $\square$

The converse also holds if $d\bar{f}_\mu > 0$ on an *open interval* around p.

Lemma 7.4.5 *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and has a local extremum at p, then either $d\bar{f}_\mu[p] = 0$ or it does not exist.*

proof If f has a local maximum at p, then there is $\delta > 0$ such that

$$f(p) = \max_{x \in (p-\delta, p+\delta)} f(x).$$

For $n > 1/\delta$, let $x_n = [p - 1/n, p + 1/n]$. Let $q_n \in x_n$ be a point where f assumes its absolute minimum on x_n. Define a sequence of intervals (y_n) by

$$y_n = \begin{cases} \text{left}(x_n) & \text{if } q_n \in \text{left}(x_n), \\ \text{right}(x_n) & \text{otherwise.} \end{cases}$$

Notice that $\mu\bar{f}(x_n) = f(p) - f(q_n) = \mu\bar{f}(y_n)$. However, each sequence (x_n) and (y_n) converges to $[p]$ in the μ topology on $\mathbf{I}\mathbb{R}$, so if $d\bar{f}_\mu[p]$ exists, then

$$d\bar{f}_\mu[p] = \lim_{n\to\infty} \frac{\mu\bar{f}(x_n)}{\mu x_n} = \lim_{n\to\infty} \frac{\mu\bar{f}(y_n)}{\mu y_n} = 2 \cdot \lim_{n\to\infty} \frac{\mu\bar{f}(x_n)}{\mu x_n} = 2 \cdot d\bar{f}_\mu[p].$$

Hence $d\bar{f}_\mu[p] = 0$. The case of a local minimum is handled by applying this argument to $-f$. $\square$

Lemma 7.4.6 *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $d\bar{f}_\mu[p] > 0$ for all $p \in (a, b)$, then $\mu\bar{f}[a, b] = |f(b) - f(a)|$.*

proof If f assumes its maximum or minimum at $p \in (a, b)$, then $d\bar{f}_\mu[p] = 0$ by Lemma 7.4.5. □

Theorem 7.4.2 *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous map on the real line and $d\bar{f}_\mu[p] > 0$ for all $p \in (a, b)$, then f is differentiable on (a, b).*

proof First we show that f is monotone on $[a, b]$ by proving it is injective. Let $f(x) = f(y)$ for points $x, y \in [a, b]$. If x and y are distinct, we may assume $x < y$. By Lemma 7.4.6,

$$\mu\bar{f}[x, y] = |f(x) - f(y)| = 0,$$

which implies f is constant on $[x, y]$. But this contradicts $d\bar{f}_\mu > 0$ on (x, y). Then $x = y$. As a continuous injection on $[a, b]$, f is either strictly increasing or strictly decreasing.

Finally, let $p \in (a, b)$. If f is increasing on (a, b), we have $\mu\bar{f}(x) = \Delta f(x)$, for all $x \in \mathbf{I}\mathbb{R}$ with $[a, b] \ll x \sqsubseteq [p]$. By Prop. 7.3.1, f is differentiable at p. Otherwise, f is decreasing, so $-f$ is increasing, and by the work above, $-f$ is differentiable at p. Again, f must be differentiable at p. □

Recall that a real valued function f defined on an open set $U \subseteq \mathbb{R}$ is said to be in $C^1(U)$ if it has a continuous first derivative on U.

Proposition 7.4.1 *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous map on the real line and $U \subseteq \mathbb{R}$ is a nonempty open set, then the following are equivalent:*

(i) *The function f belongs to $C^1(U)$.*

(ii) *The map $d\bar{f}_\mu : U \to [0, \infty)$ exists and is continuous.*

In either case, $d\bar{f}_\mu[p] = |f'(p)|$, for all $p \in U$.

proof For (ii) $\Rightarrow$ (i), we first show that f is differentiable at $p \in U$. If $d\bar{f}_\mu[p] = 0$, then $f'(p) = 0$, by Corollary 7.4.1. Otherwise, $d\bar{f}_\mu[p] > 0$, in which case the continuity of $d\bar{f}_\mu$ gives an $\varepsilon > 0$ with $(p - \varepsilon, p + \varepsilon) \subseteq U$ and

$$(\,\forall\, x \in (p - \varepsilon, p + \varepsilon)\,)\; d\bar{f}_\mu[x] > 0.$$

By Theorem 7.4.2, f is differentiable at p. For the continuity of f', let $p \in U$. If $f'(p) = 0$, then

$$|f'(x) - f'(p)| = |d\bar{f}_\mu[x] - d\bar{f}_\mu[p]|,$$

which implies f' is continuous at p. Otherwise, $f'(p) \neq 0$. In this case, we appeal to the continuity of $d\bar{f}_\mu$ and obtain an open interval $p \in (a, b) \subseteq U$ on which $d\bar{f}_\mu$ is positive. But then $f' \neq 0$ on (a, b), and so we have that either $f' > 0$ on (a, b), or $f' < 0$ on (a, b), as derivatives *always* have the intermediate value property on open intervals. Thus,

$$f'(x) = \frac{f'(p)}{|f'(p)|} \cdot d\bar{f}_\mu[x],$$

for all $x \in (a, b)$, which proves that f' is continuous at p. □

With the reassurance that $df_\mu(p)$ makes sense for a map $f : (D, \mu) \to (D, \mu)$, we may now extend the definition to mappings $f : (D, \mu) \to (E, \lambda)$ between different domains.

Definition 7.4.3 Let (D, μ) and (E, λ) be domains with measurements $\mu \to \sigma_X$ and λ. If $f : D \to E$ is a partial map and $p \in \mathrm{dom}(f) \cap X$ is not isolated in $\mathrm{dom}(f)$, then

$$df_\mu^\lambda(p) := \frac{d(\lambda f)}{d\mu}(p)$$

is called the derivative of f at p if it exists.

With two measurements around, we usually write the derivative as $df(p)$.

Example 7.4.2 If $f : X \to Y$ is a Lipschitz mapping between complete metric spaces X and Y with Lipschitz constant c, it extends to a map

$$\bar{f} : (\mathbf{B}X, \pi_x) \to (\mathbf{B}Y, \pi_y)$$

$$\bar{f}(x, r) = (fx, c \cdot r)$$

Its derivative is

$$d\bar{f}(p) = \lim_{x \to p} \frac{\pi_y \bar{f}(x) - \pi_y \bar{f}(p)}{\pi_x x - \pi_x p} = \lim_{x \to p} \frac{c \cdot (\pi_x x - \pi_x p)}{\pi_x x - \pi_x p} = c.$$

The informatic derivative reveals something important in the last example: Extensions on the formal ball model contain a small amount of information about the maps they represent. As an example, consider the case of the real line, where we have $\mathbf{I}\mathbb{R} \simeq \mathbf{B}\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz map with constant k. Write its interval domain extension as

$$f^\sharp[a, b] = f[a, b]$$

and its formal ball extension as

$$f^\flat[a, b] = \left[f\left(\frac{a+b}{2}\right) - k \cdot \frac{b-a}{2}, f\left(\frac{a+b}{2}\right) + k \cdot \frac{b-a}{2} \right].$$

The map $f^\sharp$ contains *much more* information about f than does $f^\flat$. This is easiest to see at a point p where f is differentiable:

$$df^\sharp_\mu[p] = |f'(p)| \text{ and } df^\flat_\mu[p] = k.$$

The discrepancy above is explained by one of the fundamental principles underlying measurement: The smaller the measure of an object, the larger its information content. In this case, $f^\sharp$ gives a more *precise* description of f than $f^\flat$ because $\mu f^\sharp \leq \mu f^\flat$. From the informatic viewpoint, $f^\flat$ is a linear approximation of $f^\sharp$.

Example 7.4.3 Let $[a, b] \in \mathbf{I}\mathbb{R}$ be a fixed interval. Define

$$f : \Sigma^\infty \to \mathbf{I}\mathbb{R}$$

by specifying a monotone mapping on the finite strings

$$f(x) = \begin{cases} [a, b] & \text{if } x = \varepsilon; \\ \text{left } f(s) & \text{if } x = s \cdot 0; \\ \text{right } f(s) & \text{if } x = s \cdot 1; \end{cases}$$

and then taking its unique Scott continuous extension to all of Σ^∞. It has a derivative at any infinite string $s \in \max \Sigma^\infty$,

$$df(s) = \lim_{x \to s} \frac{\mu f(x) - \mu f(s)}{\lambda x - \lambda s} = b - a,$$

which is calculated using the equality $\mu f(s) = (b - a)\lambda s$, where $\lambda x = 1/2^{|x|}$ is the probability of observing $x \in \Sigma^\infty$ and μ is the length map on $\mathbf{I}\mathbb{R}$. But does the answer make sense?

The map f is a *representation* of intervals as strings. We could, for example, treat the bisection method as an algorithm which calculates a string of $0's$ and $1's$. To do so, replace the splittings $\text{left}, \text{right} : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ with right multiplication by 0 and 1, $r_0, r_1 : \Sigma^\infty \to \Sigma^\infty$, and then instead of splitting intervals, build a string as follows: Beginning with ε, add a 0 or 1 at each iteration, according to whether or not we want to go left or right. This process ends with a string s; the interval it represents is $f(s)$.

But now we see that f is nothing more than a kind of identity map: Its informatic derivative should definitely be constant. In fact, for $\mu[a, b] = 1$, we have $df(s) = 1$. On the mathematical side, f provides a nice proof that $[0, 1]$ is the continuous image of the Cantor set.

7.5 The Analysis of Fixed Points

It often happens that partial maps on spaces have fixed points which are *unknown.* For example, the polynomial $p : \mathbb{R} \to \mathbb{R}$ given by $p(x) = x^3 + x - 1$ has has a zero on $[0, 1]$ because $p(0) \cdot p(1) < 0$. Consequently, $f(x) = x - p(x)$ has a fixed point on $[0, 1]$, even though we are not sure of what it is.

Because a partial mapping $f : X \to X$ on a space X may have an *unknown* fixed point p, methods for calculating it are important. A minimal requirement is usually that p be an *attractor.* That is, that there exist an open set $U \subseteq X$ such that for all $x \in U$, $f^n(x) \to p$. This provides a simple scheme for approximating p: Simply calculate the iterates $f^n(x)$ beginning with any $x \in U$.

For example, the unique fixed point of a contraction on a complete metric space is an attractor. In the golden section search of Example 2.4.3, the unique maximizer of a unimodal function f is also an attractor of the splitting $\max_f$ with respect to the μ topology on $\mathbf{I}\mathbb{R}$.

For a partial map $f : (D, \mu) \to (D, \mu)$ on a domain in its μ topology, the informatic derivative $df_\mu(p)$ casts light on when it is that a fixed point p is an attractor. Because of the importance of partial splittings on domains, not only in capturing the computation of fixed points for Scott continuous maps (Example 2.7.2), but also in allowing us to model computations which are not monotone, like the golden section search and the bisection method (Example 2.7.1), we begin by considering the behavior of their derivatives at fixed points.

Lemma 7.5.1 *Let $s : D \to D$ be a partial splitting which maps into* $\text{dom}(s)$. *If $s(p) = p$ and $ds_\mu(p)$ exists, then $ds_\mu(p) \leq 1$.*

proof If $x \in \text{dom}(s)$ with $x \sqsubseteq p$ and $x \neq p$, then

$$\frac{\mu s(x) - \mu s(p)}{\mu x - \mu p} = \frac{\mu s(x) - \mu p}{\mu x - \mu p} \leq \frac{\mu x - \mu p}{\mu x - \mu p} = 1,$$

where we have used $\mu x \geq \mu s(x)$ for all $x \in \text{dom}(s)$. Hence $ds_\mu(p) \leq 1$. $\square$

Then we consider partial maps f with fixed points p such that $df_\mu(p) \leq 1$. But the identity map $1 : D \to D$ has $d(1)_\mu(p) = 1$ at any element which is not compact. There is certainly no useful way to devise a theory of iteration which includes this example. Perhaps we can say something if $df_\mu(p) < 1$.

Theorem 7.5.1 *Let $f : (D, \mu) \to (D, \mu)$ be a partial map which maps into* $\text{dom}(f)$*. If $f(p) = p$ and $df_\mu(p) < 1$, then the following are equivalent:*

(i) *The partial map f is μ continuous at p.*

(ii) *There is an $a \ll p$ such that for all $x \in \text{dom}(f)$,*

$$a \sqsubseteq x \sqsubseteq p \Rightarrow \bigsqcup_{n \geq 0} f^n(x) = p.$$

In either case, we can choose $a \ll p$ so that the supremum in (ii) *is a limit in the μ topology on D.*

proof For (i) $\Rightarrow$ (ii), let $\varepsilon > 0$ be a number such that $r := df_\mu(p) + \varepsilon < 1$. By the definition of $df_\mu(p)$, there is a $\delta > 0$ such that

$$x \in \text{dom}(f) \ \& \ x \sqsubseteq p \ \& \ 0 < |\mu x - \mu p| < \delta \Rightarrow \left| \frac{\mu f(x) - \mu p}{\mu x - \mu p} - df_\mu(p) \right| < \varepsilon.$$

Without loss of generality, we can assume for this very same $\delta > 0$ that

$$x \in \text{dom}(f) \ \& \ x \sqsubseteq p \ \& \ |\mu x - \mu p| < \delta \Rightarrow f(x) \sqsubseteq f(p) = p,$$

by the μ continuity of f at p. Now let $x \in \text{dom}(f)$ be any element with $x \sqsubseteq p$ and $0 < |\mu x - \mu p| < \delta$. By the choice of δ,

$$f(x) \sqsubseteq p \ \text{ and } \ \mu f(x) - \mu p < r(\mu x - \mu p) < \delta.$$

If $\mu f(x) = \mu p$, then $f(x) = p$, by the strict monotonicity of μ. Otherwise, $0 < |\mu f(x) - \mu p| < \delta$, in which case the same reasoning applied to x, can now be applied to $f(x)$. Either way, we obtain

$$f^2(x) \sqsubseteq p \ \text{ and } \ \mu f^2(x) - \mu p < r(\mu f(x) - \mu p) < r^2(\mu x - \mu p) < \delta.$$

By induction, $f^n(x) \sqsubseteq p$ and $\mu f^n(x) - \mu p < r^n(\mu x - \mu p)$, for all $n \geq 1$. Then $\mu f^n(x) \to \mu p$ since $r^n \to 0$. By Prop. 3.1.6, $f^n(x) \to p$ in the μ topology on D. In particular,

$$\bigsqcup_{n\geq 0} f^n(x) = p.$$

In fact, this is true of any $x \in \mathrm{dom}(f)$ with $x \sqsubseteq p$ and $|\mu x - \mu p| < \delta$: If $\mu x = \mu p$, then strict monotonicity of μ gives $x = p$. Thus, it holds for all $x \in \mathrm{dom}(f) \cap \mu_\lambda(p)$ where $\lambda = \mu p + \delta$. But $\mu_\lambda(p)$ is a μ open set so there is an $a \ll p$ with $p \in \uparrow a \cap \downarrow p \subseteq \mu_\lambda(p)$.

Suprisingly, (ii) $\Rightarrow$ (i) is a trivial consequence of Lemma 7.4.1. $\square$

Theorem 7.5.1 says that *if* $df_\mu(p) < 1$ at a fixed point p where f is μ continuous, *then* p may be computed by iterating f beginning with any input x sufficiently close. In addition, by the definition of $df_\mu(p)$, *there is* a nontrivial input x with $\mu x > \mu p$ from which we can begin this iteration. This desirable property does not hold without μ continuity at p.

Example 7.5.1 The Bisection Method. From Example 2.7.1, recall that to a continuous map $f : \mathbb{R} \to \mathbb{R}$, we may assign the partial splitting

$$\mathrm{split}_f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$$

$$\mathrm{split}_f[a,b] = \begin{cases} \mathrm{left}[a,b] & \text{if } \mathrm{left}[a,b] \in C(f); \\ \mathrm{right}[a,b] & \text{otherwise;} \end{cases}$$

whose domain is $\mathrm{dom}(\mathrm{split}_f) = C(f) = \{[a,b] \in \mathbf{I}\mathbb{R} : f(a) \cdot f(b) \leq 0\}$, and whose fixed points are $\mathrm{fix}(\mathrm{split}_f) = \{[r] : f(r) = 0\}$. At any $[r] \in \mathrm{fix}(\mathrm{split}_f)$, we have

$$\frac{d(\mathrm{split}_f)}{d\mu}[r] = \frac{1}{2}.$$

But now consider the continuous function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \cdot \sin(1/x) & \text{if } x \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

The bisection split_f for this choice of f is not μ continuous at $[0]$: The elements $x_n = [-1/(2\pi n), 0] \in \mathrm{dom}(\mathrm{split}_f)$, for $n \geq 1$, form a sequence with $x_n \to [0]$ in the μ topology on $\mathbf{I}\mathbb{R}$, but

$$\mathrm{split}_f x_n = [-1/(2\pi n), -1/(4\pi n)]$$

has no limit. In addition, for each $i \geq 1$,

$$\bigsqcup_{n\geq 0} \mathrm{split}_f^n x_i = [-1/(2\pi i)],$$

which shows that each Scott open set around [0] contains an input for which the bisection method calculates a root different from 0.

A fixed point $f(p) = p$ with $df_\mu(p) < 1$ need not be maximal.

Example 7.5.2 Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (\pi/2)\sin x$. Then

$$\frac{d\bar{f}}{d\mu}[-\pi/2, \pi/2] = 0.$$

The map $\bar{f}$ is a Scott continuous map with a fixed point $p = [-\pi/2, \pi/2]$ such that $d\bar{f}_\mu(p) = 0$, but p is not a maximal element. In this case, the informatic derivative is detecting the local extrema of f.

For monotone maps with fixed points in the kernel, we have an attractor in the Scott topology.

Theorem 7.5.2 *Let $f : (D, \mu) \to (D, \mu)$ be a monotone mapping with $f(\ker \mu) \subseteq \ker \mu$. If $df_\mu(p) < 1$ at a fixed point $f(p) = p \in \ker \mu$, then there is an approximation $a \ll p$ such that*

(i) *For all $x \in D$, if $a \sqsubseteq x \sqsubseteq p$, then*

$$\bigsqcup_{n\geq 0} f^n(x) = p,$$

and this is a limit in the μ topology on D.

(ii) *The unique fixed point of f on $\uparrow\! a$ is p.*

(iii) *For all $x \in \ker \mu \cap \uparrow\! a$, $f^n(x) \to p$ in the Scott topology on $\ker \mu$.*

proof Notice that $\mathrm{dom}(f) = D$. Theorem 7.5.1 gives (i). For (ii), let $x \in \uparrow\! a$ be a fixed point of f. By monotonicity, $f^n(a) \sqsubseteq f^n(x) = x$, for all $n \geq 0$. However, by (i), we have

$$\bigsqcup_{n\geq 0} f^n(a) = p \sqsubseteq x,$$

and since $p \in \ker\mu \subseteq \max D$, $x = p$. For (iii), let $x \in \ker\mu \cap \uparrow a$. Again by monotonicity, $f^n(a) \sqsubseteq f^n(x) \in \ker\mu$, for all $n \geq 0$. But $f^n(a) \to p$ in the μ topology on D so $f^n(x) \to p$ converges in the Scott topology on $\ker\mu$. $\square$

Notice that in Theorem 7.5.2 we only need the assumption $f(\ker\mu) \subseteq \ker\mu$ to conclude (iii).

Corollary 7.5.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map on the real line with a fixed point $f(p) = p$. If $d\bar{f}_\mu[p] < 1$, then there is an $\varepsilon > 0$ such that*

$$(\,\forall\, x \in (p - \varepsilon, p + \varepsilon)\,)\; f^n(x) \to p.$$

In particular, this holds if f is differentiable at p and $|f'(p)| < 1$.

The local uniqueness of the fixed point p in Theorem 7.5.2 requires more than μ continuity at p.

Example 7.5.3 Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous map

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0; \\ 0 & \text{otherwise.} \end{cases}$$

The bisection method split_f is μ continuous at $[0]$ and $d(\text{split}_f)_\mu[0] = 1/2$, but there is no Scott open set containing $[0]$ on which $[0]$ is the only fixed point of split_f.

As an application of Theorem 7.5.2, we will prove the correctness of Newton's Method *without* using Taylor's Theorem.

Example 7.5.4 Let $f : [a, b] \to \mathbb{R}$ be a continuous function with a zero $r \in (a, b)$. If f' is nonzero and continuous on $[a, b]$ and $f''(r)$ exists, we consider the continuous map $I_f : [a, b] \to \mathbb{R}$, given by

$$I_f(x) = x - \frac{f(x)}{f'(x)}.$$

It is easy to see that $I_f(r) = r$. By extending I_f to the real line in any way whatsoever, we appeal to Theorem 7.4.1 and obtain

$$\frac{d\bar{I}_f}{d\mu}[r] = 0.$$

By Corollary 7.5.1, we see that there is an $\varepsilon > 0$ such that $I_f(x) \to r$ for all $x \in (r - \varepsilon, r + \varepsilon)$.

But what is achieved by avoiding Taylor's theorem? First, let's remind ourselves of what this classic result says.

Theorem 7.5.3 (Taylor's Theorem) *Let $p \in [a,b]$. If the first $n \geq 1$ derivatives of $f : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and $f^{(n+1)}$ exists on (a,b), then for all $x \in [a,b]$, there is a point c between x and p such that*

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(p)}{i!}(x-p)^i + \frac{f^{(n+1)}(c)}{(n+1)!}(x-p)^{n+1}.$$

Thus, to prove the correctness of Newton's method using Taylor's theorem, we must assume that f'' exists on an *open interval* containing the zero r. The proof we gave in Example 7.5.4 assumes only that $f''(r)$ exists. This gives one definite advantage to using Theorem 7.5.2 in place of Taylor's theorem: We can prove that Newton's method works on a larger class of functions.

Of course, once we know that an iterative process works *correctly,* the next question inevitably concerns the *rate* at which it works.

7.6 Rates of Convergence

In classical numerical analysis, the efficiency of an iterative algorithm is determined by calculating its *order of convergence.*

Definition 7.6.1 Let (x_n) be a sequence of reals with $x_n \to p$. If

$$0 < \lim_{n\to\infty} \frac{|x_{n+1}-p|}{|x_n - p|^{\alpha}} = r < \infty,$$

for some $\alpha \geq 1$, then α is called the *order of convergence* of the sequence. If $\alpha = 1$ then r is called the *rate of convergence* of (x_n).

In this definition, the sequence (x_n) is generated by a numerical algorithm designed to calculate p. The larger that α is, the quicker the convergence of (x_n) to p, the better the algorithm.

If $\alpha = 1$, the algorithm is said to converge *linearly.* For $\alpha = 2$, the convergence is *quadratic.* Two linearly convergent algorithms may be compared based on their rates of convergence.

Notice that orders of convergence are calculated using the uncertainty $|x_n - p|$. To extend the idea to the setting of domains with measurements, we consider sequences (x_n) which converge to their suprema p in the μ topology on D, and replace $|x_n - p|$ with $|\mu x_n - \mu p|$.

Definition 7.6.2 Let D be a domain with a measurement $\mu \to \sigma_X$. If (x_n) is a sequence in D which converges to its supremum $p \in X$ in the μ topology and

$$0 < \lim_{n\to\infty} \frac{\mu x_{n+1} - \mu p}{(\mu x_n - \mu p)^\alpha} = r < \infty,$$

for some $\alpha \geq 1$, then α is called the *order of convergence* of the sequence. If $\alpha = 1$ then r is called the *rate of convergence* of (x_n).

Recall that an *increasing* sequence (x_n) converges to its supremum p in the μ topology. We begin with linear processes: The informatic derivative enables the systematic computation of rates of convergence.

Lemma 7.6.1 *Let $s : (D, \mu) \to (D, \mu)$ be a partial map which maps into* dom(s) *and has a fixed point $p = \bigsqcup s^n x$ in the μ topology. If $ds_\mu(p)$ exists, then*

$$\lim_{n\to\infty} \frac{\mu s^{n+1}(x) - \mu p}{\mu s^n(x) - \mu p} = \frac{ds}{d\mu}(p),$$

provided $\mu s^n(x) - \mu p > 0$ for all $n \geq 0$.

proof If (x_n) is any sequence in dom$(s) \setminus \{p\}$ with $x_n \to p$ in the μ topology on D, then by the definition of $ds_\mu(p)$,

$$\frac{ds}{d\mu}(p) = \lim_{n\to\infty} \frac{\mu s(x_n) - \mu s(p)}{\mu x_n - \mu p}.$$

Finally, $s(p) = p$, and $x_n = s^n(x) \to p$ in the μ topology. $\Box$

Thus, to find the rate at which a linear algorithm s converges to a fixed point p, we find its derivative at p. But why is this a measure of efficiency?

Proposition 7.6.1 *Let $s : (D, \mu) \to (D, \mu)$ be a partial map which maps into* dom(s)*. If s is μ continuous at a fixed point p and $0 < ds_\mu(p) < 1$, then for all $0 < \varepsilon < 1 - ds_\mu(p)$, there is an $a \ll p$ such that for all $x \in$* dom(s)*,*

$$a \sqsubseteq x \sqsubseteq p \text{ and } n \geq \frac{\log(\varepsilon/(\mu x - \mu p))}{\log\left(ds_\mu(p) + \varepsilon\right)} \;\Rightarrow\; s^n x \sqsubseteq p \text{ and } |\mu s^n x - \mu p| < \varepsilon,$$

provided $x \neq p$ and $n \geq 1$.

proof As in the proof of Theorem 7.5.1, there is $\delta > 0$ such that for all $x \in \mathrm{dom}(f)$, if $x \sqsubseteq p$ and $0 < |\mu x - \mu p| < \delta$, then

$$s(x) \sqsubseteq p \text{ and } \left| \frac{\mu s(x) - \mu p}{\mu x - \mu p} - ds_\mu(p) \right| < \varepsilon,$$

provided $x \neq p$. In addition, we also saw that for any such $x \in \mathrm{dom}(s)$,

$$\mu s^n(x) - \mu p < (ds_\mu(p) + \varepsilon)^n (\mu x - \mu p) \text{ and } s^n(x) \sqsubseteq p,$$

for all $n \geq 1$. If n is a positive integer with

$$n \geq \frac{\log(\varepsilon/(\mu x - \mu p))}{\log\left(ds_\mu(p) + \varepsilon\right)}$$

then $(ds_\mu(p) + \varepsilon)^n (\mu x - \mu p) \leq \varepsilon$. For $n \geq 1$, this gives $\mu s^n(x) - \mu p < \varepsilon$.

Finally, just as in the proof of Theorem 7.5.1, we can choose $a \ll p$ such that $p \in \uparrow\!a \cap \downarrow\!p \subseteq \mu_\lambda(p)$ for $\lambda = \mu p + \delta$. $\square$

Prop. 7.6.1 gives an upper bound on the number of iterations a linear process must do before it achieves ε accuracy. In order that this estimate hold, the input x must be sufficiently close. However, even in the presence of this mathematical annoyance, we can still use it to understand why rate of convergence is a measure of efficiency.

Suppose we have two linear processes s, t which have a common fixed point p and that $0 < ds_\mu(p) < dt_\mu(p) < 1$. Let $\varepsilon > 0$. Imagine we have different inputs for s and t which both have measure λ and that $\lambda - \mu p > \varepsilon$. (If $\lambda - \mu p \leq \varepsilon$, each process is already ε close.) Then

$$\frac{\log(\varepsilon/(\lambda - \mu p))}{\log\left(ds_\mu(p) + \varepsilon\right)} < \frac{\log(\varepsilon/(\lambda - \mu p))}{\log\left(dt_\mu(p) + \varepsilon\right)},$$

that is, the number of iterations which ensure t is ε close to p also guarantee that s is ε close to p. However, it may be that s can achieve ε accuracy with fewer iterations than t. Roughly speaking, s is a better algorithm than t for calculating p.

The estimate on the number of iterations in Prop. 7.6.1 is useful because of its generality. However, we often encounter linear processes which satisfy $\mu s(x) - \mu s(p) \leq (ds_\mu(p))(\mu x - \mu p)$ for $x \sqsubseteq p$. In this case, we use the estimate

$$n \geq \frac{\log\left(\varepsilon/(\mu x - \mu p)\right)}{\log ds_\mu(p)}.$$

One question which springs to mind is: How can we know the values of μp and $ds_\mu(p)$ when p itself is unknown? Though we cannot always calculate these quantities independent of p, the estimates given for the number of iterations are still useful for comparing processes, as we saw above. On the other hand, in the case of Newton's Method, we actually know *a priori* that $\mu p = ds_\mu(p) = 0$. We can also calculate these quantities independent of p for the bisection method, the golden section search and for contraction mappings on complete metric spaces.

Example 7.6.1 As seen in Example 7.5.1, for a continuous map $f : \mathbb{R} \to \mathbb{R}$, the bisection method is captured by the partial splitting

$$\text{split}_f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$$

and the data

- $\text{dom}(\text{split}_f) = C(f) = \{[a, b] \in \mathbf{I}\mathbb{R} : f(a) \cdot f(b) \leq 0\}$
- $\text{fix}(\text{split}_f) = \{[r] : f(r) = 0\}$
- $d(\text{split}_f)_\mu[r] = 1/2$ for all $[r] \in \text{fix}(\text{split}_f)$

If r is an isolated zero of f, then split_f is μ continuous at the associated fixed point $[r]$. By the remarks following Prop. 7.6.1, if r is an isolated zero of f and $x \in C(f)$ is a sufficiently small input around r, then

$$\text{split}_f^n x \text{ for } n \geq \frac{\log(\varepsilon/\mu x)}{\log(1/2)}$$

is an ε-approximation of r.

The estimate for the number of iterations given in the last example can fail without μ continuity. If we take $f(x) = x \cdot \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$, as was considered in Example 7.5.1, then there are arbitrarily small intervals $\bar{x} \in C(f)$ with $\bar{x} \sqsubseteq [0]$, but for which $\text{split}_f \bar{x} \not\sqsubseteq [0]$. Beginning with *any* one of these intervals as input, and then doing $n \geq \log(\varepsilon/\mu x)/\log(1/2)$ iterations of split_f, leaves an interval of length $< \varepsilon$. The problem is that we are now on track to calculate a *different* zero $[r]$, rather than the one we *intended* to calculate, $[0]$.

The point is this: An estimate for the number of iterations is of little use if we do not know what we are calculating. This is why zeroes are normally assumed isolated in numerical analysis, as in Newton's method, where we assume $f'(r) \neq 0$. Thus, we expect iterative numerical methods to be μ continuous at fixed points when realized as partial maps on domains.

Example 7.6.2 The Golden Section Search. In Example 2.4.3, given a function $f : \mathbb{R} \to \mathbb{R}$ and a constant $1/2 < r < 1$, we defined the splitting

$$\max_f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$$

$$\max_f[a,b] = \begin{cases} [l(a,b), b] & \text{if } f(l(a,b)) < f(r(a,b)), \\ [a, r(a,b)] & \text{otherwise.} \end{cases}$$

where $l(a,b) = (b-a)(1-r) + a$ and $r(a,b) = (b-a)r + a$.

If f is unimodal on $[a,b]$ and its unique maximizer is $x^* \in \text{int}[a,b]$, then $\max_f$ is μ continuous at $[x^*]$ because

$$[a,b] \ll \bar{x} \sqsubseteq [x^*] \Rightarrow \max_f \bar{x} \sqsubseteq [x^*],$$

which was shown in Example 2.4.3, and because it has a derivative at $[x^*]$, given by

$$\frac{d(\max_f)}{d\mu}[x^*] = r.$$

Thus, if f is unimodal on $[a,b]$ and $x^* \in \text{int}[a,b]$, then

$$\max_f^n[a,b] \text{ for } n \geq \frac{\log(\varepsilon/(b-a))}{\log(r)}$$

is an ε-approximation of x^*.

Example 7.6.3 Contraction maps. If $f : X \to X$ is a contraction on a complete metric space (X,d) with constant $0 < c < 1$, its extension to the formal ball model

$$\bar{f} : \mathbf{B}X \to \mathbf{B}X, \quad \bar{f}(x,r) = (fx, c \cdot r)$$

has derivative $d\bar{f}_\pi(p) = c$, for all $p \in \mathbf{B}X$, as shown in Example 7.4.2. The map $\bar{f}$ is Scott continuous and hence μ continuous at all points. If we take any $x \in X$ and $r \geq d(x, fx)/(1-c)$, then

$$\bar{f}^n(x,r) \text{ for } n \geq \frac{\log(r/\varepsilon)}{\log c}$$

is an ε-approximation of the unique attractor of f.

The presence of informatic linearity in the last three examples enables us to use the estimate mentioned after Prop. 7.6.1. The next example is more interesting.

Example 7.6.4 The Regula Falsi Method. For a function $f : [a, b] \to \mathbb{R}$ such that

(i) $f(a) < 0$ and $f(b) > 0$,

(ii) $f'(x) > 0$ for all $x \in [a, b]$, and

(iii) $f''(x) \geq 0$ for all $x \in [a, b]$,

we define the partial mapping

$$r_f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$$

$$r_f[x, b] = \left[b - f(b) \left(\frac{b - x}{f(b) - f(x)} \right), b \right]$$

whose domain is

$$\text{dom}(r_f) = \{[x, b] : a \leq x \leq r\}$$

where $r \in (a, b)$ is the unique zero of f on $[a, b]$.

The map r_f is a Scott continuous splitting which maps the dcpo dom(r_f) into itself. For if $a \leq x \leq y \leq r$, we have the string of inequalities

$$a \leq x \leq b - f(b) \left(\frac{b - x}{f(b) - f(x)} \right) \leq b - f(b) \left(\frac{b - y}{f(b) - f(y)} \right) \leq r,$$

where the second follows from $f(x) \leq 0$, and the last two follow from

$$\frac{f(b) - f(x)}{b - x} \leq \frac{f(b) - f(y)}{b - y} \leq \frac{f(b) - f(r)}{b - r},$$

which is a consequence of the fact that f' is nondecreasing. This proves that r_f is a monotone splitting which takes dom(r_f) into itself. Finally, r_f is Scott continuous because its measure is Scott continuous.

By Proposition 2.7.1, if $\bar{x} \in \text{dom}(r_f)$, then

$$\bigsqcup_{n \geq 0} r_f^n(\bar{x}) \in \text{fix}(r_f),$$

but it is easy to see that fix$(r_f) = \{[r, b]\}$. Thus, iterating r_f is an algorithm for approximating r. It is called the *Regula Falsi method.* But how efficient is it?

To answer this question, we calculate the informatic derivative of r_f at the fixed point $[r, b]$ as follows:

$$\begin{aligned}
\frac{dr_f}{d\mu}[r,b] &= \lim_{\bar{x}\to[r,b]} \frac{\mu r_f(\bar{x}) - \mu r_f[r,b]}{\mu\bar{x} - \mu[r,b]} \\
&= \lim_{x\to r^-} \frac{f(b)(r-x) + f(x)(b-r)}{(f(b)-f(x))(r-x)} \\
&= \lim_{x\to r^-} \left[\frac{f(b)}{f(b)-f(x)} + \frac{f(x)-f(r)}{r-x} \cdot \frac{b-r}{f(b)-f(x)}\right] \\
&= \frac{f(b)}{f(b)-f(r)} + (-1)f'(r) \cdot \frac{b-r}{f(b)-f(r)} \\
&= 1 - \frac{f'(r)(b-r)}{f(b)}.
\end{aligned}$$

By Lemma 7.4.2, this derivative is nonnegative, and hence a number in the interval $[0, 1)$. In fact, we can see that

$$d(r_f)_\mu[r,b] \to 0 \quad \text{as} \quad b \to r$$

so the efficiency of this algorithm is determined by the closeness of b to r. Notice that it does *not* depend on a.

Once we have the derivatives of two different algorithms which solve the same problem, we can compare them to understand their respective strengths and weaknesses.

Example 7.6.5 The Bisection versus Regula Falsi. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous map and $[a, b]$ is an interval such that $f(a) < 0$ and $f(b) > 0$, $f' > 0$ on $[a, b]$ and $f'' \geq 0$ on $[a, b]$, then

$$\bigsqcup_{n\geq 0} \text{split}_f^n[a,b] = [r] \text{ and } \bigsqcup_{n\geq 0} r_f^n[a,b] = [r,b]$$

are both schemes for calculating the unique zero r of f on $[a, b]$. But which one is better? We consider two examples.

If $f(x) = x^2 - x - 1$ and $[a, b] = [1, 2]$, then $r = (1 + \sqrt{5})/2$. Thus,

$$d(\text{split}_f)[r] = \frac{1}{2} \text{ and } d(r_f)[r,b] = \frac{7-3\sqrt{5}}{2} \approx 0.145898,$$

which means that *eventually* $\mu r_f(x) - \mu[r, b] \approx 0.14(\mu x - \mu[r, b])$, as compared to $\mu\,\text{split}_f(x) - \mu[r] = 0.5(\mu x - \mu[r])$ for the bisection. In other words, *eventually* the Regula Falsi method reduces the uncertainty in an interval by about 86%, while for the bisection we know uncertainty is always reduced by 50%. This suggests that r_f is preferable in this case. In fact, if we do six iterations of each, we find that

$$\text{split}_f^6[1,2] = [1.59375, 1.625] \text{ and } r_f^6[1,2] \approx [1.618025, 2].$$

The approximation of r offered by the bisection is the midpoint of $\text{split}_f^6[1,2]$, 1.609375, while the approximation given by the Regula Falsi method is the left endpoint of $r_f^6[1,2]$, around 1.618025. Thus, the Regula-Falsi method is accurate to four decimal places, while the bisection is only accurate to one. This supports the intuition offered by the informatic derivatives calculated above: r_f converges faster than split_f in this case.

If $f(x) = x^6 - x - 1$ and $[a,b] = [1,2]$, then $r \approx 1.13472$. The informatic derivatives in this case are

$$d(\text{split}_f)[r] = \frac{1}{2} \text{ and } d(r_f)[r,b] \approx 0.85407,$$

which suggests that now it is split_f which converges faster. If we do sixteen iterations of each, we find that

$$\text{split}_f^{16}[1,2] \approx [1.134719, 1.134735] \text{ and } r_f^{16}[1,2] \approx [1.121308, 2].$$

Thus, the bisection gives the approximation $r \approx 1.13472$, while the Regula Falsi method is only accurate to one decimal place. In fact, it is only after 68 iterations that the Regula Falsi method can duplicate what the bisection achieves in 16:

$$r_f^{68}[1,2] \approx [1.13472, 2].$$

Evidently, the intuition imparted by the informatic derivative is correct in this instance as well.

We now have a uniform method for calculating rates of convergence of linear processes: Simply take the informatic derivative of a map on a domain at a fixed point. This extends what is done is numerical analysis, enabling a unified treatment not previously possible. For instance, the golden section search and the bisection method are iterative processes which have no classical descriptions as differentiable functions on the real line. Nevertheless, we have seen that they may be naturally described as mappings on domains which possess informatic derivatives. With all of this said, we still have not yet given an informatic account of higher order convergence. This is the subject of the next section.

7.7 Orders of Convergence

In the last section, we focused on the important case of a linear process – one whose order of convergence is $\alpha = 1$. We now consider processes with an order of convergence $\alpha > 1$.

Definition 7.7.1 Let D be a continuous dcpo with a measurement $\mu \to \sigma_X$. If $f : D \to \mathbb{R}$ is a partial map and $p \in \mathrm{dom}(f) \cap X$ is not isolated in $\mathrm{dom}(f)$, we define

$$f^{(0)}(p) := f(p),$$

and for $n \geq 0$, if $f^{(i)}(p)$ exists, for all $0 \leq i \leq n$, we set

$$\Delta^n f : \mathrm{dom}(f) \to \mathbb{R}$$

$$\Delta^n f(x) = f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(p)}{i!}(\mu x - \mu p)^i$$

and

$$f^{(n+1)}(p) := (n+1)! \lim_{x \to p} \frac{\Delta^n f(x)}{(\mu x - \mu p)^{n+1}},$$

provided this limit exists in the relative μ topology on $\mathrm{dom}(f)$ and the usual topology on $\mathbb{R}$.

Definition 7.7.2 Let (D, μ) be a domain with a measurement $\mu \to \sigma_X$. If $f : D \to D$ is a partial map and $p \in \mathrm{dom}(f) \cap X$ is not isolated in $\mathrm{dom}(f)$, then we define

$$d^n f_\mu(p) := (\mu f)^{(n)}(p),$$

for $n \geq 0$, provided that this limit exists in the relative μ topology on $\mathrm{dom}(f)$ and the usual topology on $\mathbb{R}$.

Higher order derivatives for mappings between different domains may be defined similarly. Notice that the definition of $d^n f_\mu(p)$ includes $df_\mu(p)$ as a special case since

$$d^1 f_\mu(p) = (\mu f)^{(1)}(p) = \lim_{x \to p} \frac{\mu f(x) - \mu f(p)}{(\mu x - \mu p)} = df_\mu(p).$$

Higher order informatic derivatives enable the systematic computation of orders of convergence.

Lemma 7.7.1 *Let* $s : (D, \mu) \to (D, \mu)$ *be a partial map which maps into* $\mathrm{dom}(s)$ *and has a fixed point* $p = \bigsqcup s^k x$ *in the* μ *topology. If* $d^i s_\mu(p) = 0$ *for* $1 \le i < n$ *and* $d^n s_\mu(p)$ *exists, then*

$$0 \le \lim_{k\to\infty} \frac{\mu s^{k+1}(x) - \mu p}{(\mu s^k(x) - \mu p)^n} = \frac{d^n s_\mu(p)}{n!},$$

provided $\mu s^k(x) - \mu p > 0$ *for all* $k \ge 0$.

proof By the definition of $d^n s_\mu(p)$ for $n \ge 1$,

$$d^n s_\mu(p) = (\mu s)^n(p) = n! \lim_{x\to p} \frac{\mu s(x) - \mu p}{(\mu x - \mu p)^n}.$$

Because $s^k x \to p$ in the μ topology on D and $\mu s^k(x) - \mu p > 0$, the claim above holds. □

Thus, the order of convergence of a process $s : D \to D$ is the order of its first nonzero informatic derivative. For example, if $ds_\mu(p) \neq 0$, s is linear; if $ds_\mu(p) = 0$ and $d^2 s_\mu(p) \neq 0$, s is quadratic; and so on.

The work of the last section establishes that this idea successfully captures the notion of a linear process: Not only does it include linear processes which have no classical descriptions as functions on the real line, like the bisection method and the r-section search, it also includes those which do (i.e., processes whose rates of convergence are calculable by a classical derivative). In order to understand the extent to which the idea captures *higher order processes,* we resume our consideration of iterative processes on the real line.

In general, if $f : \mathbb{R} \to \mathbb{R}$ is an iterative process which has a fixed point $p = f(p)$ and is better than linear ($f'(p) = 0$), then the natural way to say that it has order of convergence $n > 1$ is to require that

$$\lim_{x\to p} \frac{|f(x) - f(p)|}{|x - p|^n} \neq 0.$$

The reason for this is simple. The condition above implies that there is an $\varepsilon > 0$ such that for all $x_0 \in (p - \varepsilon, p + \varepsilon)$, we have $x_k = f^k(x_0) \to p$ and

$$\lim_{k\to\infty} \frac{|x_{k+1} - p|}{|x_k - p|^n} = \lim_{x\to p} \frac{|f(x) - f(p)|}{|x - p|^n} \neq 0.$$

That is, for all sufficiently close inputs x_0, the sequence (x_k) of iterates has order of convergence n.

In numerical analysis, the typical view is that f has order of convergence $n > 1$ if $f^{(i)}(p) = 0$ for $1 \leq i < n$ and $f^{(n)}(p) \neq 0$. The rationale for this is that if $f^{(n)}$ is continuous on an open set around p, then

$$f^{(n)}(p) = n! \cdot \lim_{x \to p} \frac{f(x) - f(p)}{(x-p)^n} \neq 0,$$

and now the analysis indicated earlier follows easily. However, if f has order of convergence n in the computation of p, it does *not* follow that $f^{(n)}(p)$ exists, *even if* $f^{(i)}(p) = 0$ for $1 \leq i < n$.

Example 7.7.1 Let I_f be the Newton iterate for the map $f(x) = x|x| + x$. Then

$$I_f'(x) = \frac{2x^2 + 2|x|}{(2|x|+1)^2},$$

and at the fixed point $r = 0$, we have $I_f'(r) = 0$. This process is quadratic:

$$\lim_{x \to r} \frac{|I_f(x) - r|}{|x-r|^2} = \lim_{x \to 0} \left(\frac{x^2}{2|x|+1} \cdot \frac{1}{x^2} \right) = 1.$$

However, $I_f''(r)$ does not exist. In fact, neither does

$$\lim_{x \to r} \frac{I_f(x) - r}{(x-r)^2}.$$

Interestingly, $f''(r)$ also fails to exist, even though $I_f'(r) = 0$.

Thus, the classical derivative is not general enough to support a proper definition of order of convergence. The problem is that the classical derivative is analytic in nature, closely and perhaps irreversibly tied to geometric intuitions. The rates of change that take place during the execution of an algorithm are informatic. To make this clear, we first develop a simple view of iterative processes that emphasizes the way they manipulate information.

Definition 7.7.3 An *iterative process* on the real line is a partial mapping $f : \mathbb{R} \to \mathbb{R}$ defined on an open set around its unique fixed point $p = f(p)$.

Definition 7.7.4 The *order of convergence* of an iterative process on the real line f is the unique $\alpha \geq 1$ such that

$$\lim_{x \to p} \frac{|f(x) - p|}{|x-p|^\alpha} \neq 0$$

provided that such an α exists.

Definition 7.7.5 Let $f : \mathbb{R} \to \mathbb{R}$ be an iterative process on the real line with $p = f(p)$. For $x \in \mathbb{R}$, set $x^\star = [\min\{x, p\}, \max\{x, p\}]$, and then define a partial map

$$f^\star : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$$
$$x^\star \mapsto f(x)^\star$$

for all $x \in \mathrm{dom}(f)$.

We understand $\star$ as an informatic operator. A typical input to f is a real number $x \in \mathrm{dom}(f)$. Because f is an algorithm for computing $p = f(p)$, the information content of x is $|x - p|$, which means the object of information corresponding to x is $x^\star$. Similarly, the object of information corresponding to f is $f^\star$. But what does this teach us about f?

If we view f as a process which manipulates real numbers, we cannot necessarily use the classical derivative to calculate its order of convergence, as seen in Example 7.7.1. However, if we adopt the view that f is really $f^\star$, a process that manipulates information, we find that the *informatic* derivative can *always* be used to calculate its order of convergence.

Theorem 7.7.1 *For an iterative process $f : \mathbb{R} \to \mathbb{R}$ on the real line with $p = f(p)$ and an integer $n \geq 1$,*

$$\lim_{x \to p} \frac{|f(x) - p|}{|x - p|^n} \text{ exists}$$

if and only if

$$d^n f^\star_\mu[p] \text{ exists and } d^i f^\star_\mu[p] = 0, \text{ for } 0 \leq i < n.$$

In either case,

$$d^n f^\star_\mu[p] = n! \cdot \lim_{x \to p} \frac{|f(x) - p|}{|x - p|^n}$$

proof ($\Rightarrow$) The proof is by induction. First the base case $n = 1$. We have $d^0 f^\star_\mu[p] = \mu f^\star[p] = \mu[p] = 0$ and

$$df^\star_\mu[p] = \lim_{x^\star \to [p]} \frac{\mu f(x)^\star}{\mu x^\star} = \lim_{x^\star \to [p]} \frac{|f(x) - p|}{|x - p|} = \lim_{x \to p} \frac{|f(x) - p|}{|x - p|},$$

where the last equality follows from $0 < \mu x^\star < \delta \Leftrightarrow 0 < |x - p| < \delta$. Now assume the claim holds for $n \geq 1$. To prove it for $n + 1$, set

$$\varphi_m = \lim_{x \to p} \frac{|f(x) - p|}{|x - p|^m}$$

and notice that $\varphi_i = 0$ for $0 \leq i \leq n$, while $\varphi_{n+1} \in \mathbb{R}$. By assumption, $d^i f^\star_\mu[p] = 0$ for $i < n$ and $d^n f^\star_\mu[p]$ exists. Then by the definition of $d^n f^\star_\mu[p]$,

$$d^n f^\star_\mu[p] = n! \cdot \lim_{x^\star \to [p]} \frac{\mu f(x)^\star}{(\mu x^\star)^n} = n! \cdot \varphi_n = 0.$$

Thus, if $d^{n+1} f^\star_\mu[p]$ exists, it is given by

$$d^{n+1} f^\star_\mu[p] = (n+1)! \cdot \lim_{x^\star \to [p]} \frac{\mu f(x)^\star}{(\mu x^\star)^{n+1}},$$

but this derivative clearly exists, as it is equal to $(n+1)! \cdot \varphi_{n+1}$.
($\Leftarrow$) Let $n \geq 1$. Then by the definition of $d^n f^\star_\mu[p]$, we have

$$d^n f^\star_\mu[p] = n! \cdot \lim_{x^\star \to [p]} \frac{\mu f(x)^\star}{(\mu x^\star)^n} = n! \cdot \lim_{x \to p} \frac{|f(x) - p|}{|x - p|^n},$$

which guarantees that the limit in question exists. $\square$

Corollary 7.7.1 *For an iterative process $f : \mathbb{R} \to \mathbb{R}$ on the real line with $p = f(p)$ and an integer $n \geq 1$, the following are equivalent:*

(i) *For all i with $0 \leq i < n$, $d^i f^\star_\mu[p] = 0$, and $d^n f^\star_\mu[p] \neq 0$.*

(ii) *The order of convergence of f is n.*

Newton's method is a process which *always* has a second informatic derivative, but *need not* possess a classical second derivative.

Proposition 7.7.1 *Let $f : [a, b] \to \mathbb{R}$ be a continuous map with a zero $r \in (a, b)$ such that*

(i) *$f' \neq 0$ on $[a, b]$, and*

(ii) *f'' exists on (a, b) and is continuous at r.*

Then for the Newton iterate $I_f : [a, b] \to \mathbb{R}$, we have $d(I^\star_f)_\mu[r] = 0$ and

$$d^2 (I^\star_f)_\mu[r] = \left| \frac{f''(r)}{f'(r)} \right|,$$

even though I_f need not have a classical second derivative at r.

proof Let $x \in (a,b)$. By Taylor's theorem, for all $t \in [a,b]$, there is a point c between x and t such that

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(c)}{2!}(t-x)^2.$$

Then setting $t = r$ gives

$$\left(x - \frac{f(x)}{f'(x)}\right) - r = \frac{f''(c)}{2f'(x)}(x-r)^2.$$

Thus, for all $x \in (a,b)$, there is a c between x and r such that

$$\frac{I_f(x) - I_f(r)}{(x-r)^2} = \frac{f''(c)}{2f'(x)}.$$

By the continuity of f' and f'' at r,

$$\lim_{x \to r} \frac{I_f(x) - I_f(r)}{(x-r)^2} = \frac{f''(r)}{2f'(r)}.$$

The result is now immediate by Theorem 7.7.1. $\square$

In Example 7.7.1, we saw that if $f(x) = x|x| + x$, then $d^2(I_f^\star)_\mu[r] = 2 \cdot 1$ even though $I_f''(r)$ does not exist. However, $f''(r)$ does not exist, so f does not provide an example of the sort mentioned in Prop. 7.7.1.

Example 7.7.2 Let $f(x) = x + x^5 \sin(1/x)$ with $f(0) = 0$. Then f has a continuous second derivative and $f' > 0$ around $r = 0$. By Proposition 7.7.1,

$$d(I_f^\star)_\mu[r] = d^2(I_f^\star)_\mu[r] = 0.$$

However, $I_f''(r)$ does not exist.

Thus, if one adopts the view that an iterative process is a partial map on a domain, there is a uniform method available for calculating its order of convergence: Find the order of its first nonzero informatic derivative.

In closing, we should emphasize the importance of using compact representations of iterative processes when taking informatic derivatives. For example, $I_f^\star$ is a compact representation of Newton's method, while $\bar{I}_f$ is not. In short, use partial mappings whose domains include only those elements that may *actually* be used in the computation process. The wrong choice of representation can not only prevent an informatic derivative from existing, but can also limit the effectiveness of the theory. Higher order derivatives seem especially sensitive to this. The questions at the end of this chapter may shed some light on this issue.

7.8 Affine Mappings on the Real Line

In this section, we consider an application of the informatic derivative very different from those in the previous: An informatic characterization of affine mappings on the real line.

Definition 7.8.1 A mapping $f : \mathbb{R} \to \mathbb{R}$ is *affine* if there are constants $a, b \in \mathbb{R}$ such that $f(x) = ax + b$, for all $x \in \mathbb{R}$.

As pointed out after Example 7.4.2, if $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz mapping with Lipschitz constant m, we have what are in general two distinct extensions of f to the interval domain. The first is of course the canonical extension

$$\bar{f} : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R} :: [a, b] \mapsto f[a, b].$$

We call it canonical because given any other monotone extension f^* of f to $\mathbf{I}\mathbb{R}$, we have $f^* \sqsubseteq \bar{f}$. The second is $f^\flat : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ given by

$$f^\flat[a, b] = \left[f\left(\frac{a+b}{2}\right) - m \cdot \frac{b-a}{2}, f\left(\frac{a+b}{2}\right) + m \cdot \frac{b-a}{2} \right].$$

The measure of both $\bar{f}$ and $f^\flat$ is bounded above by $m\mu$. In the case of $f^\flat$, we actually have the equality $\mu f^\flat = m\mu$.

Theorem 7.8.1 *For a function $f : \mathbb{R} \to \mathbb{R}$, the following are equivalent:*

(i) *The mapping f is affine.*

(ii) *There is a number $m \geq 0$ such that f has a unique monotone extension $\bar{f} : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ satisfying $\mu \bar{f} \leq m\mu$.*

proof (i) $\Rightarrow$ (ii): Let $f(x) = ax + b$ be an affine map and set $m = |a|$. Then the canonical extension $\bar{f}$ is Scott continuous and $\mu\bar{f}(x) = m\mu(x)$, for all $x \in \mathbf{I}\mathbb{R}$. If $f^* : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ is any monotone extension of f with $\mu f^* \leq m\mu$, then $f^* \sqsubseteq \bar{f}$. Then $m\mu \geq \mu f^* \geq \mu\bar{f} = m\mu$. Thus,

$$f^* \sqsubseteq \bar{f} \text{ and } \mu f^* = \mu\bar{f} \;\Rightarrow\; f^* = \bar{f},$$

where we apply the strict monotonicity of μ for each $x \in \mathbf{I}\mathbb{R}$.

(ii) $\Rightarrow$ (i): Let $m \geq 0$ and $f^* : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ be the unique monotone extension of f which satisfies $\mu f^* \leq m\mu$.

First we claim that $|f(b) - f(a)| \leq m|b - a|$, for all $a, b \in \mathbb{R}$. Let $\bar{x} = [\min\{a, b\}, \max\{a, b\}]$. Then $\bar{x} \sqsubseteq [a], [b]$ and so by monotonicity, $f^*(\bar{x}) \sqsubseteq f^*[a], f^*[b]$. But f^* extends f, so $f^*[a] = [f(a)]$ and $f^*[b] = [f(b)]$. Thus, $f(a), f(b) \in f^*(\bar{x})$, which gives

$$|f(b) - f(a)| \leq \mu f^*(\bar{x}) \leq m\mu\bar{x} = m|b - a|,$$

proving that f is a Lipschitz map.

Because f is Lipschitz, it is continuous, so the canonical extension $\bar{f}$ exists and satisfies $\mu\bar{f} \leq m\mu$. In addition, we have the extension $f^\flat$, which satisfies $\mu f^\flat = m\mu$. By uniqueness, $\bar{f} = f^* = f^\flat$, which means $\mu\bar{f} = \mu f^\flat = m\mu$. Thus, $d\bar{f}_\mu[x] = m$, for all $x \in \mathbb{R}$. By Prop. 7.4.1,

$$f \in C^1(\mathbb{R}) \text{ and } |f'(x)| = m,$$

for all $x \in \mathbb{R}$. If $m = 0$, then $f' \equiv 0$; otherwise, $m > 0$, in which case the continuity of f' implies that it is constant. Either way, f is a function with a constant derivative, which proves that it is affine. □

There are two technical points regarding Theorem 7.8.1. First, the number m is unique, and second, one can replace the word 'monotone' with 'Scott continuous' and the result still holds. Theorem 7.8.1 is a nontrivial example of the subtlety of informatic structure.

7.9 Questions

(i) Define the informatic derivative at compact elements.

(ii) Use the informatic derivative and the idea of modelling topological spaces to provide a definition of derivative for functions $f : X \to \mathbb{R}$ from a compact connected metric space into the reals. The idea should be good enough to admit a method for either optimizing real valued maps on X, or for solving systems of equations $f_i : X^n \to \mathbb{R}$ for $1 \leq i \leq n$.

(iii) If $d\bar{f}_\mu[p] > 0$ exists on $\mathbf{I}\mathbb{R}$, is f differentiable at p? If not, then consider the following problem: Let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function with a zero $f(r) = 0$. Define $I_f(x) = x - f(x)/M$ for a constant $M \neq 0$. If we can find a value of M such that

$$\frac{d\bar{I}_f}{d\mu}[r] < 1$$

then by Theorem 7.5.2, we can compute r by iterating I_f. When does this happen?

(iv) Prove the mean value theorem for the informatic derivative on a large class of domains. It reads as follows: If f is differentiable on the compact set $\uparrow x \cap \max D$, then there is $y \in \uparrow x$ and $p \in \uparrow y \cap \max D$ such that
$$\mu f(x) = \mu f(y) = df_\mu(p) \cdot \mu y.$$

(v) Show that the dimension of a fractal may be calculated using an informatic derivative on $\mathbf{C}D$. Try first for the case of *similarity dimension.* The "open set condition" introduced by Hutchinson implies that this is the same as Hausdorff dimension in a large number of cases. It should have special meaning when interpreted on the convex powerdomain.

(vi) Use the informatic derivative to analyze the Nelder-Mead Simplex Method.

(vii) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous real valued maps and assume that $d\bar{f}[p]$ and $d\bar{g}[p]$ both exist. Let $h := f + g$. Then

(a) Does $d\bar{h}[p]$ exist?

(b) Can we write $d\bar{h}[p]$ in terms of $d\bar{f}[p]$ and $d\bar{g}[p]$?

(viii) When is a function with a zero derivative constant?

(ix) Find a domain of maps $f : D \to D$ ordered according to their complexity. The idea is that $f \sqsubseteq g \Rightarrow g$ has a smaller derivative than f (that is, g is a better algorithm than f).

(x) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous selfmap and suppose that $d^2\bar{f}_\mu[x]$ exists for all $x \in \mathbb{R}$. Prove that f is an affine mapping of the real line.

(xi) A *similitude* on a complete metric space X is a map $f : X \to X$ which satisfies $d(f(x), f(y)) = m \cdot d(x, y)$, for a constant m. Use the formal ball model $(\mathbf{B}X, \pi)$ to show that Theorem 7.8.1 generalizes to similitudes on complete metric spaces.

Bibliography

[1] S. Abramsky and A. Jung. *Domain Theory.* In S. Abramsky, D. M. Gabbay, T. S. E. Maibaum, editors, Handbook of Logic in Computer Science, vol. III. Oxford University Press, 1994

[2] A. V. Arhangelskii *General Topology III: Paracompactness, Function spaces, Descriptive theory.* A. V. Arhangelskii, editor, Encyclopedia of Mathematical Sciences, vol 51, Springer-Verlag, 1995

[3] A. V. Arhangelskii. *Mappings and spaces.* Russian Math. Surveys 21 (1966), p.115–162.

[4] G. Boolos and R. Jeffrey. *Computability and Logic.* Cambridge University Press, New York, Third Edition (1989)

[5] G. Choquet. *Lectures in Analysis.* W. A. Benjamin, vol. I, New York (1969)

[6] A. Edalat. *Dynamical Systems, measures and fractals via domain theory.* Information and Computation, 120:32–48, July 1995.

[7] A. Edalat and R. Heckmann. *A Computational Model for Metric Spaces.* Theoretical Computer Science 193 (1998) 53–73 .

[8] R. Engelking. *General Topology.* Polish Scientific Publishers, Warszawa, 1977.

[9] A. H. Frink. *Distance functions and the metrization problem.* Bulletin of the American Math Society 43 (1937), p.133-142.

[10] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices.* Springer-Verlag, Berlin, 1980.

[11] G. Gruenhage. *Generalized metric spaces.* In K. Kunen, J.E. Vaughan, editors, Handbook of Set Theoretic Topology, Elsevier Science Publishers, 1984

[12] J. E. Hutchinson. *Fractals and self-similarity.* Indiana University Mathematics Journal 30 (1981), p.713–747

[13] A. Jung. *Cartesian Closed Categories of Domains.* CWI Tracts 66, Amsterdam, 1989.

[14] A. Kechris. *Classical Descriptive Set Theory.* Springer-Verlag, 1994

[15] S. Kleene. *A Note On Recursive Functions.* Bulletin of the American Math Society 42 (1936), p.544–546.

[16] J. Lawson. *Spaces of Maximal Points.* Mathematical Structures in Computer Science 7 (1997), p.543–555.

[17] M. Mislove *Topology, domain theory and theoretical computer science.* Topology and its Applications 89 (1998) 3–59

[18] D. Scott *Outline of a mathematical theory of computation.* Technical Monograph PRG-2, November 1970

[19] S. Willard *General Topology.* Addison-Wesley Publishing Company, 1970

[20] H. Zhang. *Dualities of domains.* PhD thesis, Department of Mathematics, Tulane University, 1993.